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Editorial Note

The following two papers had been nearly completed by V.N. Gribov before he passed away in summer 1997. Since these papers are the only documents on his ideas about quark confinement in QCD, the editors welcomed the proposal to publish the two manuscripts *post mortem* after they were edited by his wife and some of his closest friends. Even though Gribov's work on quark confinement has not led yet to a generally accepted solution of this problem – in fact some fundamental aspects are rather controversial – the publication of these unconventional and novel ideas may initiate new theoretical developments in this area. The editors consider this point a compelling reason for the publication of the two manuscripts in Eur. Phys. J. C.

Hamburg, February 26, 1999 P.M. Zerwas
J. Bartels

QCD at large and short distances* (annotated version)†

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Abstract. A formulation of QCD which contains no divergences and no renormalization procedure is presented. It contains both perturbative and nonperturbative phenomena. It is shown that, due to its asymptotically free nature, the theory is not defined uniquely. The chiral symmetry breaking and the nature of the octet of pseudoscalar particles as quasi-Goldstone states are analyzed in the theory with massless and massive quarks. The U(1) problem is discussed.

1 Introduction

In this paper I show how to formulate an asymptotically free theory in such a way that it includes perturbative and nonperturbative phenomena simultaneously.

The idea is the following. Contrary to an infrared free theory, in an asymptotically free theory, the divergences prevent us from writing even perturbation expansions in a unique, well-defined way. We can, however, make use of the fact that divergences in the theory occur only in the Green's functions and the vertices. On the other hand, knowing the Green's functions and the vertices, we can express all the other amplitudes through them perturbatively in a unique way. Thus, we have to formulate equations for Green's functions and vertices in a form that does not contain any divergences. If this is done, the solutions of these equations will contain both perturbative and nonperturbative phenomena.

To avoid technical complications, in Sect. 2 of the paper we will derive these equations in an Abelian theory, in which usual perturbation theory is also well defined. In

* This work was completed in Germany under the Humboldt Research Award Program.

† The original version of this paper, completed by the author in April 1997, was submitted to the hep-ph archive (hep-ph/9708424) a few days after Prof. V. N. Gribov passed away on August 13, 1997.
This is the first of two papers concluding his 20-year study of the problem of quark confinement in QCD. This annotated version is the result of an attempt by a group of his colleagues to understand the paper, starting in November 1997 in Orsay. A number of misprints were eliminated, most of the equations were checked, and some corrected. Comments have been added in order to make the text easier to read. These comments are displayed in square brackets. Many theorists participated in the process; the comments were assembled, and the final version prepared, by Yu. Dokshitzer, C. Ewerz, A. Kaidalov, A. Mueller, J. Nyiri and A. Vainshtein.

Sect. 3, we generalize the equations for a non-Abelian theory and discuss the connection between perturbative and nonperturbative effects. The equations have an integro-differential structure in which the asymptotic behaviour of the Green's functions is defined by the boundary conditions. The main conclusion in this section is that in the region of large momenta the Green's functions of quarks and gluons contain additional parameters compared to normal perturbation theory. These parameters can be associated with different types of "condensates". These nonperturbative parameters can be defined by solving the equations and finding nonsingular solutions in the infrared region. *A priori*, it is not clear what type of additional conditions have to be imposed on this system of equations in order to fix a nonsingular solution. For example, it can be the conservation of the axial current or of other currents that is formally satisfied from the point of view of the Lagrangian but is not ensured because of the divergences.

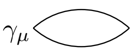
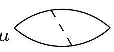
In Sect. 4, we will show that these equations make it possible, for the first time, to analyze the problem of spontaneous symmetry breaking in an asymptotically free theory and to find the approximate value of the critical coupling at which symmetry breaking occurs. Also, these equations allow us to answer the fundamental question for asymptotically free theories, namely, how the bound states – the hadrons – have to be treated in these theories and how these bound states influence the equations for the Green's functions.

The analysis leads to the following conclusion. The conditions for axial current conservation of flavour nonsinglet currents (in the limit of zero bare-quark masses) require that eight Goldstone bosons (the pseudoscalar octet) have to be regarded as elementary objects with couplings defined by Ward identities. This is so in spite of the fact that the couplings of these states to fermions decrease at large fermion virtualities.

The same analysis provides a new possibility for the solution of the U(1) problem. In this solution, the flavour singlet pseudoscalar η' is a normal bound state of $q\bar{q}$ without a point-like structure. The mass of this bound state is different from zero and can be calculated in the limit of massless quarks. For massive quarks, the pseudoscalar octet becomes massive. The masses of the pseudoscalar mesons, however, are not calculable in terms of bare-quark masses, because of logarithmic divergences, and have to be regarded as unknown parameters which, in the real case of confined quarks, are defined by the self-consistence of the solution of the infrared problem. These states have an essential influence on the equations for the Green's functions, which, as it will be shown in the next paper, can be used constructively in solving the confinement problem if the effective coupling in the infrared region is not too large. In this case, the integro-differential equations can be reduced to a system of nonlinear differential equations, and the theory looks like a theory of particle propagation in self-consistent fields defined by the Green's functions themselves (as is the case in Landau's Fermi liquid theory). The self-consistent fields are fields of gluons and π mesons.

2 Equations for Green's functions in QED

In QED, we have two Green's functions – those of the photon $D_{\mu\nu}(k)$ and electron $G(q)$ – and a vertex function $\Gamma_\mu(k,q)$. The photon Green's function is defined by the vacuum polarization operator $\Pi_{\mu\nu}(k)$, which can be expressed symbolically by a sum of Feynman diagrams

$\Pi_{\mu\nu}(k) = e_0^2 \Big\{ \gamma_\mu$  $\gamma_\nu + \gamma_\mu$  $\gamma_\nu + \dots \Big\}$. (1)

In order to obtain series not containing divergences, we will try to consider the derivatives of $\Pi_{\mu\nu}(k)$ as functions of momenta. Due to current conservation, we have

$$\Pi_{\mu\nu}(k) = (\delta_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2)\ , \qquad (2)$$

where $\Pi(k^2)$ contains only logarithmic divergences. Differentiating (2) twice, we will have

$$\partial^2 \Pi_{\mu\nu}(k) = 6\Pi(k^2)\delta_{\mu\nu} + \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(\partial_\xi + 6)\partial_\xi \Pi(k^2); \qquad (3)$$

we here use

$$\partial_\xi \equiv q_\mu \frac{\partial}{\partial q_\mu}.$$

The second term in (3) does not contain divergences; the first one does. In order to obtain a finite expression, consider

$$\partial_\mu \partial_\sigma \Pi_{\sigma\nu}(k) = -3\delta_{\mu\nu}\Pi(k^2) - 3\frac{k_\mu k_\nu}{k^2}\partial_\xi \Pi(k^2). \qquad (4)$$

From (3) and (4) it follows that the quantity

$$\begin{aligned} &\partial^2 \Pi_{\mu\nu}(k) + 2\partial_\mu \partial_\sigma \Pi_{\sigma\nu}(k) &(5)\\ &\quad = -6\frac{k_\mu k_\nu}{k^2}\partial_\xi \Pi(k^2) + \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(\partial_\xi + 6)\partial_\xi \Pi(k^2) \end{aligned}$$

does not contain divergences.

In Feynman gauge, $D_{\mu\nu}(k^2)$ is as follows:

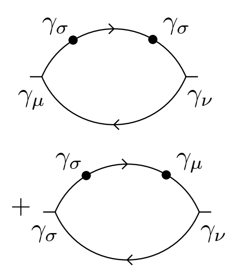
$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2(1-\Pi(k^2))} \equiv \frac{1}{k^2}\frac{e^2(k^2)}{e_0^2(k^2)}\delta_{\mu\nu}. \qquad (6)$$

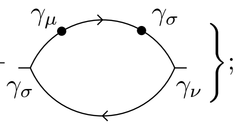
Any diagram contains only the product $e_0^2 D_{\mu\nu}(k)$, and therefore

$$\begin{aligned} e_0^2 D_{\mu\nu}(k) &= \frac{1}{k^2}e^2(k^2)\delta_{\mu\nu}, \\ \partial_\xi \Pi(k^2) &= -\partial_\xi \frac{e_0^2}{e^2}. \qquad (7) \end{aligned}$$

In first order, this means

$\partial_\xi \dfrac{1}{e^2} = \dfrac{1}{3}\dfrac{k_\mu k_\nu}{k^2} \Bigg\{$

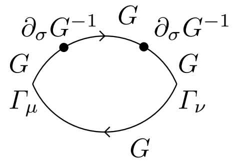


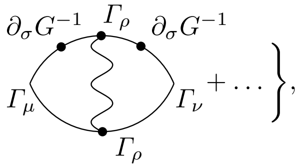
$+$ 

$\Big\}$; (8)

the integrals on the right-hand side are convergent. This has to hold in any order. Hence, we can include in (8) the exact electron Green's functions and the exact vertex functions and add all the corresponding diagrams which were not included. As a result, we can write

$$k_\mu \partial_\mu \frac{1}{e^2} = \frac{1}{3}\frac{k_\mu k_\nu}{k^2}\sum_p \Bigg\{$$



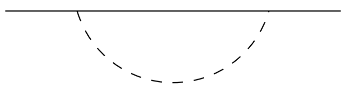
$+$ 

$+\cdots\Bigg\}$, (9)

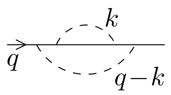
where $\sum_p$ denotes the sum over the permutations of the indices similarly to (8); a quantity e^2/k^2 corresponds to each photon line. Equation (9) is an equation for e^2 (in the form of series in e^2) provided that $G(k)$ and $\Gamma_\mu(k,q)$ are known.

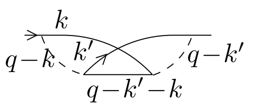
In order to obtain equations for electron Green's functions and vertex functions, we have to remember that these functions can change rapidly, even if e^2 is small, because they can have infrared singularities of the type $e^2\ln((q^2-m^2)/m^2)$ or ultraviolet singularities of the type $(\alpha/\alpha_0)^\gamma$. It has been proven that it is possible to arrange the differentiation in such a way that there will be an expansion only in e^2.

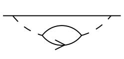
To write an equation for the fermion Green's function not containing any divergences, we have to differentiate twice the self-energy of the fermion or, equivalently, its inverse Green's function. Let us consider $\partial_{\mu\mu}G^{-1}(q)$; the diagrammatic expression for G^{-1} will be the following. The simplest diagram is


.

Diagrams of the next order are



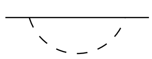
$+$ 

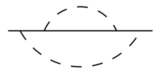
$+$ 
.

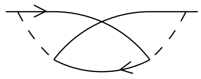
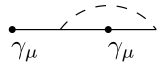
It can be easily shown that

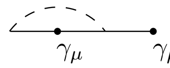
$$\partial^2 \frac{1}{k^2+\mathrm{i}\varepsilon} = -4\pi^2\mathrm{i}\delta^4(k)\ . \tag{10}$$

Using this equality, we have

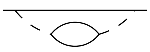
∂^2  $= -\dfrac{e_0^2}{4\pi^2}\gamma_\mu G_0\gamma_\mu = -g_0\gamma_\mu G_0\gamma_\mu,$

∂^2  $= -g_0\gamma_\mu G_1\gamma_\mu,$

∂^2  $= -g_0$ 

$-g_0$ 

$+\ \delta_2,$

∂^2  $= \gamma_\mu \displaystyle\int \frac{\mathrm{d}^4k}{4\pi^2\mathrm{i}}\, G_0(q')\,\partial^2\frac{g_1(k^2)}{k^2}\,\gamma_\mu.$

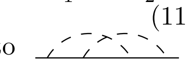
Restricting ourselves to $\partial^2 1/k^2$, we can write

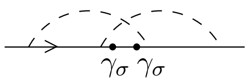
$$\partial^2\sigma_1 = -g_0\gamma_\mu G_0\gamma_\mu,$$

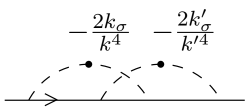
$$\begin{aligned}\partial^2\sigma_2 = &-g_0\gamma_\mu G_0\partial_\mu\sigma_1 - g_0\partial_\mu\sigma_1 G_0\gamma_\mu\\ &-g_0\gamma_\mu G_1\gamma_\mu - g_1\gamma_\mu G_0\gamma_\mu\end{aligned}$$

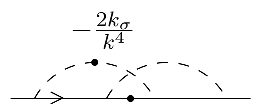
and, consequently,

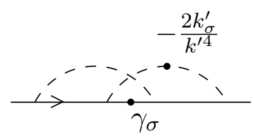
$$\partial^2 G^{-1} = g\,\partial_\mu G^{-1}\,G\,\partial_\mu G^{-1}, G^{-1} = G_0^{-1} + G_1^{-1} + G_2^{-1}\ . \tag{11}$$

The term δ_2 describes the contribution to , which does not contain overlapping divergences, and is of the form

$\delta_2 =$ 

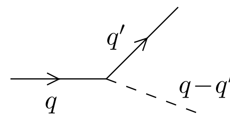
$+$ 

$+$ 

$+$ 

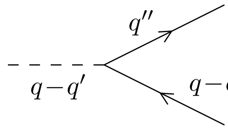
. (12)

For the calculation of higher-order diagrams, it is convenient to adopt the following principle. Beginning from the first point of interaction, we shall relate the external momentum to the photon line:



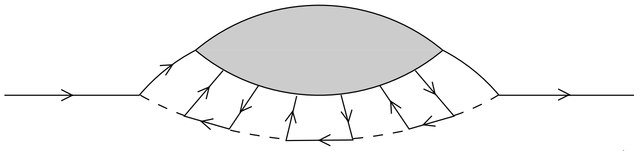
. (13)

The next photon interaction can be expressed as



. (14)

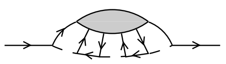
Now we relate the external momentum to the positron line. If the positron is emitting a photon, we keep sending the external momentum along the positron line up to its annihilation. As a result, we get a tree structure


, (15)

in which the initial photon line can end only at the final electron. It can be shown that every line in the diagram will be passed only once if the photons included in the fermion self-energy are not taken into account. This means that in this approach, exact electron Green's functions have to be used with bare vertices, since the idea is basically the exclusion of overlapping divergences. Taking the second derivative of one of the photon propagators and restricting ourselves to the contribution $4\pi^2\delta^4(k)g(0)$, we obtain both on the left-hand side and the right-hand side photon-emission amplitudes with zero momentum. Due to gauge invariance, however, a zero-momentum photon cannot change the state of the system (it is emitted from the external lines). The photon- emission amplitude equals $\varGamma_\mu(q,0) = \partial_\mu G^{-1}(q)$ [within this convention, the bare vertex is $-e\gamma_\mu$]. The final contribution of the differentiation has the structure (11). The above statement, which essentially means that the emission of a zero-momentum photon is determined by the total charge, can be proven by the Ward identity.

Thus, we have

$$\partial^2 G^{-1} = g(0)\partial_\mu G^{-1} G \partial_\mu G^{-1}$$

$+$  . (16)

The remaining diagrams contain first derivatives of photon and positron lines and second derivatives of positron lines. All these diagrams can be expressed in terms of the exact $\varGamma_\mu$, G and D functions. They do not contain divergences, except for those graphs which correspond to the photon self-energy.

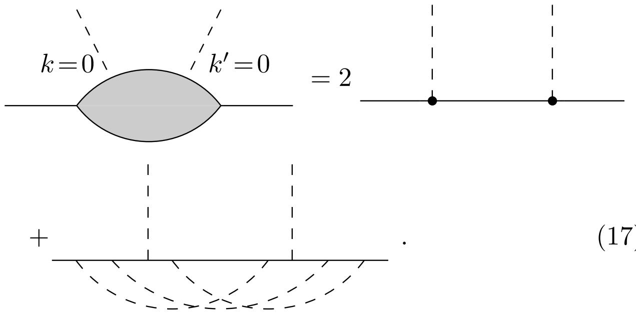
The first term in (16) has the structure

$$\frac{1}{2} g(0)\, \tilde{M}_{\nu\nu}(q, k=0)\ ,$$

where $\tilde{M}_{\nu\nu}(q, k = 0)$ is a quantity close to the Compton scattering amplitude, in the sense that they would be equal if we differentiated all fermion propagators including those inside the electron Green's function. (The factor 1/2 is present because the Compton amplitude contains the sum of diagrams with momenta k and $-k$). The Compton scattering amplitude $M_{\nu\nu}(q, k = 0)$ satisfies the Ward identity

$$M_{\nu\nu}(q) = G^{-1}\,\partial^2 G\, G^{-1} = 2\partial_\mu G^{-1}\, G\, \partial_\mu G^{-1} - \partial^2 G^{-1},$$

which has the simple diagrammatical meaning

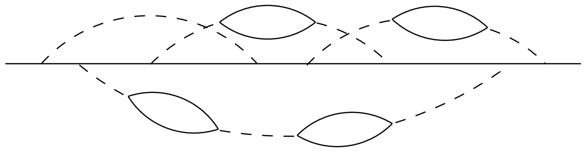


. (17)

The second term on the right-hand side of (17) represents the contribution of photons which, if we carry out the aforementioned differentiation, corresponds to photons inside the Green's function, which are not present in $\tilde{M}_{\nu\nu}$. Hence,

$$\tilde{M}_{\nu\nu}(q,0) = 2\partial_\mu G^{-1}\, G\, \partial_\mu G^{-1}.$$

All the diagrams for $\partial^2 G^{-1}$ are of the form


. (18)

The sum of the diagrams (18) is built up of the exact photon Green's functions, each of which equals (6), which equal $4\pi^2 g(k)(1/k^2)$. By differentiating the photon lines, we have calculated the contribution of $\partial^2 1/k^2$. Thus, there remains

$$\mathcal{I} = -\int \frac{\mathrm{d}^4k}{4\pi^2\mathrm{i}} \frac{\tilde{M}_{\nu\nu}(q,k)}{2}\, \partial^2 \frac{g(k)-g(0)}{k^2}\ , \qquad (19)$$

where we have introduced $g(0)$ in order to avoid a contribution from $\delta^4(k)$. This expression contains logarithmic corrections coming from the ultraviolet region ($k > q$). Replacing $g(k) - g(0)$ by $g(q) - g(0) + g(k) - g(q)$ and performing an integration by parts, we get

$$\mathcal{I} = (g(q) - g(0))\, \frac{\tilde{M}_{\nu\nu}(q,0)}{2} - \int \frac{\mathrm{d}^4k}{4\pi^2\mathrm{i}} \frac{g(k)-g(q)}{k^2}\, \partial^2 \frac{\tilde{M}_{\nu\nu}(q,k)}{2}\ . \qquad (20)$$

The first term in (20) can be explicitly calculated, while the second one does not contain any logarithms because of the presence of the difference $g(k) - g(q)$. We can rewrite (20) in the form

$$\mathcal{I} \equiv \big(g(q) - g(0)\big)\partial_\mu G^{-1} G \partial_\mu G^{-1} + \delta_1;$$

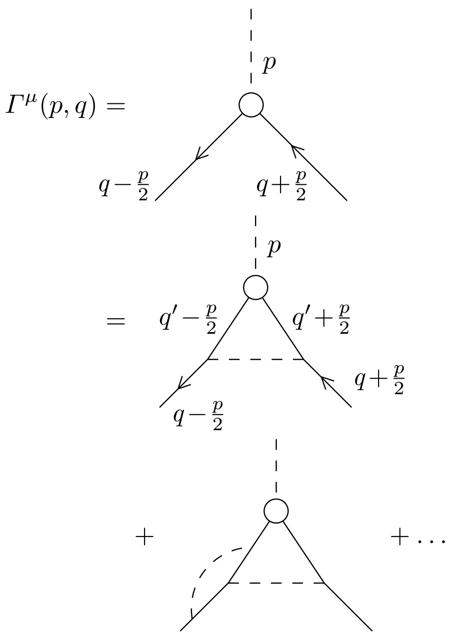
consequently, $\partial^2 G^{-1}$ can be expressed as

$$\partial^2 G^{-1}(q) = g(q)\partial_\mu G^{-1} G \partial_\mu G^{-1} + \delta_1 + \delta_2, \qquad (21)$$

where δ_1 is defined by [the second line of] (20), and δ_2 contains first-order derivatives of the photon and positron lines and second-order derivatives of the positron lines, as has been explained above. The first term contains all singularities of the types $\alpha \ln((q^2 - m^2)/m^2)$ and $(\alpha/\alpha_0)^\gamma$.

3 Equations for the vertex function and for the amplitudes of interaction with the external field

To obtain the equation for the vertex function, let us express $\Gamma_\mu(p,q)$ as a set of diagrams containing exact Green's functions:

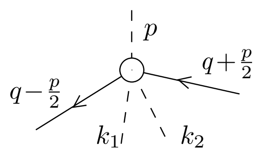


(22)

We shall relate the external momenta to the lines in the diagrams in the same way as we did when we derived the equation for the Green's function. Calculating the second-order derivative in q, we obtain

$$\partial_q^2 \Gamma^\mu(p,q) = -\,g(0)\tilde{M}^\mu_{\nu\nu}(p,q,k)\Big|_{k=0} + \dots\,, \tag{23}$$

where $\tilde{M}^\mu_{\nu\nu}(p,q,k)|_{k=0}$ is the contribution to the amplitude of the process



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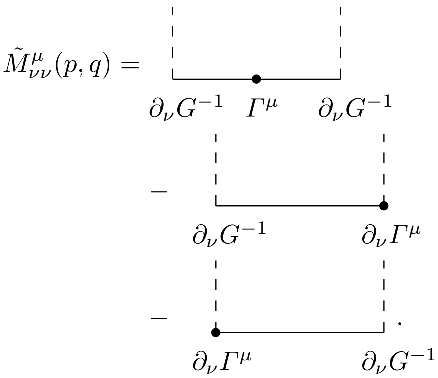
which corresponds to the emission of photons with momenta $k_1, k_2 = 0$ from the external legs:



(24)

Inserting (24) into (23) and replacing $g(0)$ by $g(q)$ we get

$$\partial^2\Gamma^\mu(p,q) = g(q)\Big\{A_\nu(q_2)\,\partial_\nu\Gamma^\mu(p,q) + \partial_\nu\Gamma^\mu(p,q)\,\tilde{A}_\nu(q_1) - A_\nu(q_2)\,\Gamma^\mu(p,q)\,\tilde{A}_\nu(q_1)\Big\} + \partial^2\tilde{\Gamma}^\mu(p,q)\,. \tag{25}$$

Here we have introduced the notations

$$q_{1,2} = q \pm \frac{p}{2}, \quad A_\mu(q) = \partial_\mu G^{-1}(q)G(q), \quad \text{and}$$
$$\tilde{A}_\mu(q) = G(q)\partial_\mu G^{-1}(q). \tag{26}$$

The correction terms $\partial^2\tilde{\Gamma}^\mu$ are defined as a set of diagrams with exact Green's functions and vertices. They contain first-order derivatives of both the photon lines and the positron lines, second-order derivatives of the positron lines and, as in the case of the Green's function, corrections due to the replacement of $g(0)$ by $g(q)$.

The same equation is valid for the interaction amplitude of fermions with the external field, provided that this interaction does not depend on the relative momentum q of the fermions.

The equations for the interactions with external fields are essential. Indeed, if these equations have solutions that decrease at large virtualities of the fermions – i.e., solutions which do not require driving terms – this means the existence of bound states.

For the sake of convenience, we rewrite the equation (25) in the form

$$\partial^2\phi(p,q) = g(q)\Big\{A_\nu(q_2)\partial_\nu\phi(p,q) + \partial_\nu\phi(p,q)\tilde{A}_\nu(q_1) - A_\nu(q_2)\phi(p,q)\tilde{A}_\nu(q_1)\Big\}\,, \tag{27}$$

which will be understood as the equation for the bound state, with a spin that is defined by the invariant structure of the matrix ϕ.

It is important to note that the accuracy of the equations for the vertex $\Gamma_\mu(p,q)$ and for the bound states differs from the accuracy of the equation for the Green's function. The functions $\Gamma_\mu(p,q)$ and $\phi(p,q)$ depend on, among other things, the ratios p^2/q_1^2, p^2/q_2^2. If these parameters become large, then Γ_μ and $\phi(p,q)$ contain, in general, the so-called Sudakov logarithms, which are not included in the equations (25) and (27). Hence, the equations are valid only if

$$\alpha \ln\frac{p^2}{q_1^2}\ln\frac{p^2}{q_2^2} < 1\,. \tag{28}$$

4 Equations for Green's functions in QCD

QCD is the theory of interacting quarks and gluons. The description of quarks is more or less the same as in QED. Gluons, however, are very different. Even the fact that a gluon has to have a multi-component wave function because it is a spin-1 particle is not seen explicitly. In the usual approach, it is described by a Green's function $D_{\mu\nu} = \langle A_\mu(x), A_\nu(y)\rangle$, which, in momentum space (in Feynman gauge), can always be written in the form

$$D_{\mu\nu} = \frac{\delta_{\mu\nu}}{k^2} C(k^2). \tag{29}$$

This equation contains only one unknown function. All spin properties of the gluon are included in the momentum dependence of its interaction. In order to introduce multi-component Green's functions, we have to formulate the theory such that the interaction is momentum-independent: We have to replace the usual description of the gluons and their interactions by a Duffin–Kemmer type formulation. In the framework of this description, the gluon Lagrangian is

$$\mathcal{L}(x) = -\{\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\nu, A_\mu]\} F_{\mu\nu} + \frac{1}{2} F_{\mu\nu} F_{\mu\nu}, \tag{30}$$

[where the trace over colour indices is implied]. The potential A_μ and the field strength $F_{\mu\nu}$ are independent quantities. The interaction is momentum-independent and equals $[A_\mu, A_\nu]F_{\mu\nu}$. In this formulation, we have three independent Green's functions

$$\langle A_\mu, A_\nu\rangle, \qquad \langle F_{\mu\nu}, A_\rho\rangle, \qquad \langle F_{\mu\nu}, F_{\rho\sigma}\rangle. \tag{31}$$

In order to fix the gauge in a covariant way, it is, of course, necessary to introduce ghosts by adding a gauge-fixing term

$$\mathcal{L}^g = \frac{\zeta}{2}(\partial_\mu A_\mu)^2. \tag{32}$$

The three Green's functions can be combined into one by introducing the ten-component state [implying $\rho < \sigma$]

$$\Psi = \begin{pmatrix} A_\nu \\ \frac{1}{m} F_{\rho\sigma} \end{pmatrix}. \tag{33}$$

We use the parameter m, which has the dimension of mass, to convert the lower component of Ψ into the same dimension as the upper component. The equations for the fields corresponding to the Lagrangians (30) and (32) are

$$\begin{aligned} &\nabla_\nu F_{\nu\mu} + \zeta \partial_\mu \partial_\nu A_\nu = 0, \\ &\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] - F_{\mu\nu} = 0. \end{aligned} \tag{34}$$

In terms of the state Ψ, we have

$$\beta_\mu (\mathrm{i}\partial_\mu + gA_\mu)\Psi - m\gamma_- \Psi - \frac{\gamma_+}{m}\zeta(\hat{p}^2 - p^2)\Psi = 0; \tag{35}$$

$$\hat{p} \equiv \mathrm{i}\beta_\mu \partial_\mu, \quad \hat{p}^2 = -\beta_\mu \beta_\nu \partial_\mu \partial_\nu, \quad p^2 = -\partial_\mu \partial_\mu.$$

In the above equations, β_μ are Duffin–Kemmer matrices satisfying the commutation relation

$$\beta_i \beta_k \beta_l + \beta_l \beta_k \beta_i = \delta_{ik}\beta_l + \delta_{kl}\beta_i\,. \tag{36}$$

These matrices connect the [four] upper Ψ^ν and [six] lower $\Psi_{\rho\sigma}$ components of the field Ψ in (33),

$$\beta_\mu = \left(\begin{array}{c|c} 0 & S_\mu \\ \hline (S_\mu)^\dagger & 0 \end{array}\right).$$

[Their nonzero components, the 4×6 matrices S_μ] have a simple representation

$$(S_\mu)^\nu_{\rho\sigma} = \mathrm{i}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}). \tag{37}$$

The quantities $\gamma_\pm$ are projection operators, projected onto the upper and lower components of Ψ .

If we want to preserve the dimension of the $\langle A_\mu, A_\nu\rangle$ component of the free Green's function, D_0 has to satisfy the equation

$$\left[\frac{\hat{p}}{m} - \gamma_- - \frac{\gamma_+}{m^2}\zeta(\hat{p}^2 - p^2)\right] D_0 = \frac{1}{m^2}. \tag{38}$$

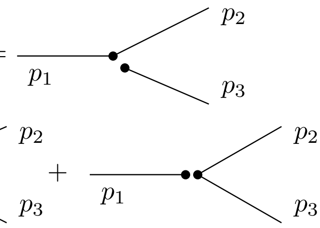
The solution of this equation is

$$D_0 = \frac{1}{p^2}\left[\frac{\hat{p}}{m} + C_1\gamma_+ + \frac{\gamma_-}{m^2}(\hat{p}^2 - p^2)\right]. \tag{39}$$

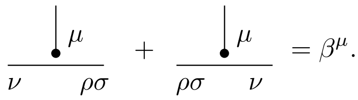
In Feynman gauge, $\zeta = 1$ and $C_1 = 1$, and in Landau gauge, $C_1 = \hat{p}^2/p^2$. The three terms in (39) correspond to the three independent Green's functions (31).

The vertex for the interaction of three gluons with momenta p_1, p_2, p_3, colours a, b, c, and Duffin–Kemmer indices α, β, γ has the structure

$\Gamma_{a,\alpha,p_1;b,\beta,p_2;c,\gamma,p_3} =$



(40)



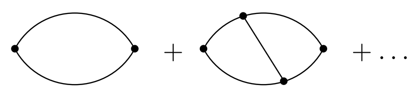
The coupling constant remains in our notation g.

Let us consider in this approach the properties of the exact Green's function

$$D^{-1} = \hat{k} - \gamma_- - \gamma_+\,\zeta(\hat{k}^2 - k^2) - \Sigma - \delta\zeta(\hat{k}^2 - k^2), \tag{41}$$

$$\hat{k} = \frac{\hat{p}}{m},$$

where Σ is defined diagrammatically:

$\Sigma =$  (42)

It contains four matrix elements Σ_{++}, Σ_{-+}, Σ_{+-}, Σ_{--}. It can be also represented in the form

$$\Sigma = \hat{k}\Sigma_1 + \gamma_-\Sigma_2 + \gamma_+\hat{k}^2\Sigma_3. \tag{43}$$

The factor $\hat{k}^2$ in the third term is necessary to preserve the current conservation; in the first order, the ghost contributes only to Σ_3.

We added the term $\delta\zeta$ in (41) so that we could fix the gauge for the exact Green's function. Instead of (41), we can write

$$D^{-1} = Z_1^{-1}\hat{k} - Z_2^{-1}\gamma_- - Z_3^{-1}\hat{k}^2\gamma_+ - \zeta(\hat{k}^2 - k^2). \quad (44)$$

The Green's function then will be

$$D = \frac{1}{(Z_1^{-2} - Z_2^{-1}Z_3^{-1})p^2}\{Z_1^{-1}\hat{k} + Z_2^{-1}C\gamma_+ + Z_2Z_1^{-2}\hat{k}^2\gamma_-\} + \frac{Z_2\gamma_-}{m^2}, \quad (45)$$

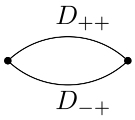
where

$$C = 1 + \left(\frac{\hat{k}^2}{k^2} - 1\right)\left[\frac{Z_2Z_1^{-2} - Z_3^{-1}}{\zeta} - 1\right]. \quad (46)$$

In Feynman gauge,

$$\zeta = Z_2Z_1^{-2} - Z_3^{-1}. \quad (47)$$

In order to understand the meaning of the denominator in (45), let us consider the renormalization properties of simple diagrams, for example $\Sigma_{+-} + \Sigma_{-+}$:

$\Sigma_{+-} + \Sigma_{-+} = g_0^2\Gamma_{+-+}$  $\Gamma_{++-} + (+ \to -)$ (48)

where $g_0 m$ is the effective bare coupling [we expect; see (56) below]:

$$\Gamma_{+-+} \sim Z_1^{-1}. \quad (49)$$

If this is true, we have in Feynman gauge

$$\Sigma_{+-} + \Sigma_{-+} \sim \frac{Z_1^{-3}Z_2^{-1}}{(Z_1^{-2} - Z_2^{-1}Z_3^{-1})^2}. \quad (50)$$

However, $p_\mu\partial_\mu\Sigma_{-+}$ [in the manuscript, Σ_{--}] has to be proportional to αZ_1^{-1}. This means that we have to expect

$$\frac{Z_1^{-3}Z_2^{-1}}{(Z_1^{-2} - Z_2^{-1}Z_3^{-1})^2} \equiv \frac{\alpha(p)}{\alpha_0}Z_1^{-1}, \quad (51)$$

i.e.,

$$Z_2Z_1^{-2} - Z_3^{-1} = \sqrt{\frac{\alpha_0}{\alpha}Z_1^{-2}Z_2}\,.$$

This is our definition of $\alpha(p)$. It has all known properties of the renormalized coupling α. As a result, D can be written in the form

$$D = \sqrt{\frac{\alpha(p^2)}{\alpha_0}Z_1^2Z_2^{-1}}\left\{Z_2Z_1^{-1}\hat{k} + C\gamma_+ + (Z_2Z_1^{-1}\hat{k})^2\gamma_-\right\}\frac{1}{p^2} + \frac{Z_2\gamma_-}{m^2}\,. \quad (52)$$

In this approach, α_0 is not a quantity coming from the normalization; it is the bare coupling. The theory has to be defined as the limit $\alpha_0 \to 0$. In this context, the expression (52) has a very interesting property. In the limit $\alpha_0 \to 0$, the Green's function D contains only Z_1^{-1} and Z_2^{-1}. According to (51), in this limit, $Z_3^{-1} = Z_1^{-2}Z_2$. Because of this, when we regard the equations for Z_1^{-1} and Z_2^{-1} as we did for the fermionic Green's function in QED, α_0 has to disappear. Consequently, we have equations for Z_1^{-1} and Z_2^{-1}, with $\alpha(p^2)$ being arbitrary. The equation for Z_3^{-1} will not help, for due to the equality $Z_3^{-1} = Z_1^{-2}Z_2$, it has to be an identity. We will see that an equation for α appears when we will consider the correction of the order of $\sqrt{\alpha_0}$.

Before formulating the equation for the Green's function, let us see what the Ward identity looks like in this formulation. The bare vertex is

$$\Gamma_\mu^0 = \hat{f}\beta_\mu$$

[with $\hat{f}$ the colour matrix (the structure constant)]. Consider the relation between $(p_2 - p_1)_\mu\Gamma^\mu(p_2, p_1)$ and the Green's function:

$$\begin{aligned} p_\mu \cdot \beta_\mu = \hat{p}_2 - \hat{p}_1 &= m\left[\hat{k}_2 - \gamma_- - (\hat{k}_1 - \gamma_-)\right] \\ &= m[D_0^{-1}(k_2) - D_0^{-1}(k_1)] \\ &\quad + m\,\gamma_+\zeta_0[(\hat{k}_2^2 - k_2^2) - (\hat{k}_1^2 - k_1^2)]. \end{aligned} \quad (53)$$

Hence, at $p \to 0$ [hereafter, $\partial_\mu = \partial/\partial p_\mu$],

$$\Gamma_\mu^0|_{p=0} = \hat{f}\left[m\partial_\mu D_0^{-1} + m\,\gamma_+\zeta_0\partial_\mu(\hat{k}^2 - k^2)\right], \quad (54)$$

which is, of course, the usual complication due to the Slavnov–Taylor Ward identity. But in this formalism, the additional term is

$$-m\partial_\mu\left(\zeta_0\frac{\partial}{\partial\zeta_0}D_0^{-1}\right).$$

Inserting this vertex into an arbitrary diagram, we obtain [for the full vertex] the relation

$$\Gamma_\mu = \hat{f}m\left[\partial_\mu D^{-1} - \zeta_0\frac{\partial}{\partial\zeta_0}\partial_\mu D^{-1}\right]. \quad (55)$$

According to (44) and (47), in Feynman gauge we can write [$\partial_\zeta \equiv \zeta_0\partial/\partial\zeta_0$]:

$$\Gamma_\mu = \hat{f}m\partial_\mu\left[(Z_1^{-1} - \partial_\zeta Z_1^{-1})\hat{k} - \gamma_-(Z_2^{-1} - \partial_\zeta Z_2^{-1}) - \gamma_+\hat{k}^2(Z_1^{-2}Z_2 - \partial_\zeta(Z_1^{-2}Z_2))\right]. \quad (56)$$

The quantities $\partial_\zeta Z_1^{-1}$, $\partial_\zeta Z_2^{-1}$ must be calculated from the equations for Z_1^{-1} and Z_2^{-1}. But if the equations are formulated in terms of the exact Green's functions, the dependence on ζ enters only through the Green's functions D which, according to (45) and (46), include ζ only through the quantity C defined in (46). In the limit $\alpha_0 \to 0$, however, C does not depend on ζ. Consequently, $\partial_\zeta Z_1^{-1}$ and

$\partial_\zeta Z_2^{-1}$ are equal to zero, and we have the following simple relation for the vertex at zero momentum:

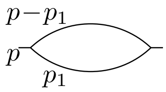
$$\Gamma_\mu = \hat{f}\, m\, \partial_\mu \tilde{D}^{-1}. \tag{57}$$

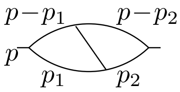
Here $\tilde{D}^{-1}$ is defined by (56); it does not contain gauge-fixing terms:

$$\tilde{D}^{-1} = Z_1^{-1}\hat{k} - \gamma_- Z_2^{-1} - \gamma_+ Z_1^{-2} Z_2\, \hat{k}^2. \tag{58}$$

Let us first consider the equations for Z_1^{-1} and Z_2^{-1} in the same way as we did for fermions. As in the case of QED, we will use the Feynman gauge in order to simplify the one-gluon contribution to the equation for the Green's function.

$$\partial^2(Z_1^{-1}\hat{k} - Z_2^{-1}\gamma_-) =$$

$= \partial^2\Big\{$ 

$+$ 

$+ \ldots \Big\}$ (59)

Each line in (59) contains the exact propagator (52). Differentiating the propagator along the upper line, we obtain the contribution corresponding to the second derivative of one line,

$$\partial^2 D = -4\pi^2 \mathrm{i}\delta^4(p)\gamma_+ Z^{\frac{1}{2}}(0) - \frac{4p_\mu}{p^4}\partial_\mu\left(NZ^{\frac{1}{2}}\right) + \frac{1}{p^2}\partial^2\left(NZ^{\frac{1}{2}}\right) + \partial^2\frac{Z_2\gamma_-}{m^2}, \tag{60}$$

where

$$Z = Z_1^2 Z_2^{-1}\frac{\alpha(0)}{\alpha_0}, \tag{61}$$

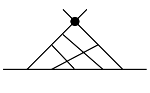
and we introduce the notation

$$D = \frac{Z^{\frac{1}{2}}}{p^2}N + \frac{Z_2\gamma_-}{m^2}. \tag{62}$$

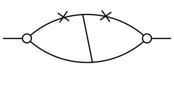
The contribution of this derivative to the equation will have the form

$$-4\pi^2 \mathrm{i} Z^{\frac{1}{2}} M(p,p) + M_1. \tag{63}$$

Here $M(p,p)$ is the gluon–gluon scattering amplitude at zero momentum of one of the gluons. The second term, M_1, is defined diagrammatically:

$M_1 =$  .

Taking the first derivative of two different lines, we get the contribution

$M_2 =$  $+ \ldots .$ (64)

As a result, we have

$$\partial^2(Z_1^{-1}\hat{k} - Z_2^{-1}\gamma_-) = \frac{\alpha_0}{\pi} Z^{\frac{1}{2}} M(p,p) + M_1 + M_2. \tag{65}$$

We have to remember that on the right-hand side we must take only the matrix elements $\langle -|+\rangle$, $\langle +|-\rangle$ and $\langle -|-\rangle$. Writing

$$M(p,p) \;=\; \Gamma^\mu\, D\, \Gamma^\mu + \tilde{M}(p,p), \tag{66}$$

and using the Ward identity (57), we obtain an equation of the same structure as that for fermions:

$$\partial^2(Z_1^{-1}\hat{k} - Z_2^{-1}\gamma_-) = \hat{f}^2\frac{\alpha_0}{\pi}\frac{Z^{\frac{1}{2}}(0)Z^{\frac{1}{2}}(p)}{k^2}\,\partial_\mu\tilde{D}^{-1} \times\left(N + \frac{Z_2 Z^{-\frac{1}{2}}}{m^2}\gamma_-\right)\partial_\mu\tilde{D}^{-1} + L; \tag{67}$$

$$L = \tilde{M} + M_1 + M_2\ , \quad \hat{f}^2 = N_c = 3.$$

If we now use the same trick as before, namely, we replace $Z(0)$ by $Z(p)$ and redefine L, we will have, in the limit $\alpha_0 \to 0$,

$$\partial^2 Z_1^{-1}[\gamma_-, \hat{k}] = 3\frac{\alpha(p)}{\pi\, k^2}\left[\gamma_-, \partial_\mu\tilde{D}^{-1}\left(Z_1\hat{k} + Z_1^2 Z_2^{-1}\gamma_+ + Z_2\hat{k}^2\gamma_-\right)\partial_\mu\tilde{D}^{-1}\right] + [\gamma_-, L'], \tag{68}$$

$$\partial^2 Z_2^{-1}\gamma_- = 3\frac{\alpha(p)}{\pi\, k^2}\,\gamma_-\partial_\mu\tilde{D}^{-1}\left(Z_1\hat{k} + Z_1^2 Z_2^{-1}\gamma_+ + Z_2\hat{k}^2\gamma_-\right)\partial_\mu\tilde{D}^{-1}\gamma_- + \gamma_- L'\gamma_-. \tag{69}$$

4.1 [Equation for quark Green's function]

The equation for the fermionic Green's function in QCD will differ from (21) only by the factor $\lambda^a\lambda^a = 4/3$ (λ^a are colour matrices). The reason is the following. The derivation of the equation (21) was based on the relation between the fermionic Green's function and the amplitude of zero-momentum photon emission by a fermion

$$\Gamma_\mu(q,0) = \partial_\mu G^{-1}(q); \tag{70}$$

in the usual formulation, this relation is not correct in QCD. The simple relation (57) for the amplitude of the zero-momentum gluon emission by a gluon implies, however, that the amplitude of a zero-momentum gluon emission by a quark has to be

$$\Gamma_\mu^a(q,0) = \lambda^a (Z_1^{-2}Z_2)^{\frac{1}{4}}\partial_\mu G^{-1}(q). \tag{71}$$

Together with (52), this leads to the equation (21).

4.2 [Equations for vertices]

The equation for the colourless vertices remains also the same.

The equation for a three-gluon vertex, however, will be essentially different. In the same way as for the fermionic case, but taking into account the noncommutativity of the gluon coupling, we can show that

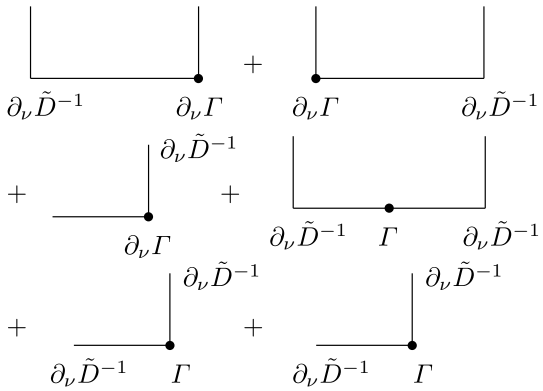
$$\Gamma_{\nu\mu\rho\sigma} = {}_\nu(\Gamma^\mu)_{\sigma\rho} = \hat{\Gamma}^\mu$$

satisfies the following equation:

$$\partial^2 \hat{\Gamma}^\mu \left(q+\frac{p}{2}, p, q-\frac{p}{2}\right)$$
$$= \frac{3\alpha}{\pi}\Big\{ A_\nu(p_1)\frac{\partial \hat{\Gamma}^\mu}{\partial q_\nu} + \frac{\partial \hat{\Gamma}^\mu}{\partial q_\nu}\tilde{A}_\nu(p_3) + (A_\nu(p))_{\mu\mu'}\frac{\partial \hat{\Gamma}^{\mu'}}{\partial p_\nu}$$
$$-A_\nu(p_1)\hat{\Gamma}^\mu \tilde{A}_\nu(p_3) - A_\nu(p_1)\hat{\Gamma}^{\mu'}(\tilde{A}_\nu(p))_{\mu'\mu}$$
$$-(A_\nu(p))_{\mu\mu'}\hat{\Gamma}^{\mu'}\tilde{A}_\nu(p_3)\Big\}; \qquad (72)$$

$$A_\nu = \partial_\nu \tilde{D}^{-1} D \quad , \quad \tilde{A}_\nu = D\partial_\nu \tilde{D}^{-1}\,;$$
$$p_1 = q + \frac{p}{2} \quad , \quad p_3 = q - \frac{p}{2}\,.$$

The right-hand side corresponds to all possible gluon emissions from the external lines:



Higher-order terms can be written in the same diagrammatic way.

The fourth term in (72) and in the corresponding diagrammatic expression equals zero at $\alpha = 0$, since $\tilde{D}\Gamma\tilde{D} = 0$ at $\alpha = 0$.

The equation (72) is, indeed, quite different from the equation for a colourless vertex. The main difference comes from the fact that on the right-hand side, it contains derivatives of Γ not only over q but also over p. Hence, in order to find Γ, we have to write three different equations for second derivatives over three different momenta.

A similar equation is valid for the quark–gluon vertex.

4.3 [Equations and boundary conditions]

Knowing the Green's function and the vertices, one can write all the other amplitudes for the interactions and quarks and gluons in the usual perturbative way. These amplitudes have no divergences and contain, inside the gluon Green's function, the unknown function $\alpha(p)$. To formulate the theory in an unambiguous way, without any references to the cutoff and the regularization, we have to find the equation for $\alpha(p)$ as we did in QED, and learn how to write the higher terms more elegantly and constructively. I postpone the investigation of this problem for another paper. We will now discuss the main difference between an asymptotically free theory and an infrared free theory.

The equations for Green's functions of quarks and gluons are proven to be second-order integro-differential equations. To solve them, we need boundary conditions. In an infrared theory, the boundary conditions are known: They are the conditions for the existence of free elementary particles at small momenta. In an asymptotically free theory, the interaction is small at large momenta, and we expect to have here a perturbative solution. However, in the region of large momenta, all the equations have two types of solutions: the perturbative solution and solutions which decrease as some power of the momenta and therefore contain dimensional parameters reflecting the density of different condensates. These parameters have to be defined either by the introduction of additional conditions of the type of conservation laws or by the self-consistency of the solutions in the small-momentum region (or both). The next section of this paper will be devoted, essentially, to the discussion of this problem.

5 Spontaneous symmetry breaking in asymptotically free theories

In this section we will consider the equation for the fermion Green's function in QCD,

$$\partial^2 G^{-1}(q) = g(q)\partial_\mu G^{-1}(q) G(q) \partial_\mu G^{-1}(q)\,, \qquad (73)$$

where

$$g(q) = \frac{\alpha(q^2)}{\pi}\lambda^a\lambda^a = \frac{4\alpha(q^2)}{3\pi}.$$

This equation has been discussed extensively in connection with the problem of quark confinement [1]. Here we shall use the equation for the discussion of the spontaneous breaking of chiral symmetry in asymptotically free theories. The asymptotic freedom is reflected in this equation by the fact that $\alpha(q^2)$ decreases when $q^2 \to \infty$. We will assume that in the limit $q \to 0$, $\alpha(q^2)$ approaches a finite value.

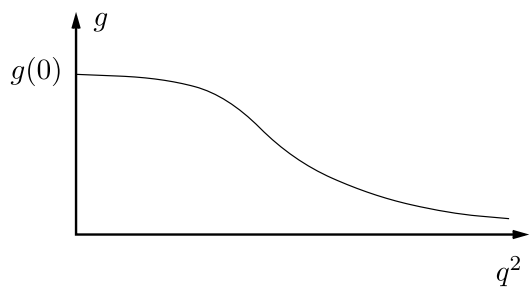


Fig. 1.

At $q^2 \to \infty$ the solution of the equation (73) has the form

$$G^{-1}(q) = Z^{-1}\left[(m - \hat{q}) + \frac{\nu_1^3}{q^2} + \frac{\nu_2^4\hat{q}}{q^4}\right], \qquad (74)$$

$$\hat{q} = \gamma_\mu q_\mu\,.$$

If $\alpha = 0$, the quantities Z, m, ν_1, and ν_2 are arbitrary parameters. In the case when $\alpha(q^2)$ is defined by perturbation theory, Z, m, ν_1, and ν_2 are the following functions:

$$Z = Z_0 \left(\frac{\alpha}{\alpha_0}\right)^{\gamma_Z}, \quad m = m_0 \left(\frac{\alpha}{\alpha_0}\right)^{\gamma_m},$$
$$\nu_{1,2} = \nu_{1,2}^0 \left(\frac{\alpha}{\alpha_0}\right)^{\gamma_{1,2}}. \tag{75}$$

The anomalous dimensions γ_Z, γ_m and $\gamma_{1,2}$ can easily be found from (73). Generally speaking, the solution depends on four parameters. In the limit $q^2 \to \infty$, the chiral-invariant solution can be written as

$$G^{-1} = -Z^{-1}\hat{q}\left(1 - \frac{\nu_2^4}{q^4}\right). \tag{76}$$

The general solution (74) corresponds to massive quarks. In the solution that corresponds to spontaneously broken chiral symmetry, $m_0 = 0$. In this case the mass term decreases when $q^2 \to \infty$; the term ν_1 is responsible for the violation of the symmetry.

Multiplying (73) by $G(q)G^{-1}(q)$, we obtain the equation

$$\partial^2 G^{-1}(q) = gA_\mu(q)A_\mu(q)G^{-1}(q) \tag{77}$$

where

$$A_\mu(q) = \partial_\mu G^{-1}(q)G(q). \tag{78}$$

Clearly, (77) has a structure which corresponds to the equation for particle propagation in the self-consistent field $gA_\mu A_\mu$. It is easily seen that (73) is equivalent to the equation for A_μ of the form

$$\partial_\mu A_\mu = -\beta A_\mu A_\mu, \qquad \beta = 1 - g. \tag{79}$$

The matrix G^{-1} is defined by two invariant functions, and can be written as[1]

$$G^{-1} = Z^{-1}(q)[m(q) - \hat{q}] \equiv \rho\, \mathrm{e}^{-\hat{n}\frac{\phi}{2}} \tag{80}$$

where

$$\hat{n} = \frac{\hat{q}}{q}, \quad q = \sqrt{q^2}.$$

Here A_μ is

$$A_\mu = \frac{\partial_\mu \rho}{\rho} - \frac{1}{2}\hat{n}\,\partial_\mu\phi - \partial_\mu\hat{n}\,\sinh\frac{\phi}{2}\,\mathrm{e}^{\hat{n}\frac{\phi}{2}}. \tag{81}$$

Inserting (80) and (81) into (77), we obtain a set of nonlinear equations for ρ and ϕ. We can linearize the equation for ρ by writing

$$\rho = \left(\frac{u}{q}\right)^{\frac{1}{\beta}} \tag{82}$$

for a constant β, or

$$\rho = \frac{u}{q}\exp\left\{\int \frac{g}{\beta}\left(\frac{\dot{u}}{u} - 1\right)\frac{\mathrm{d}q}{q}\right\} \tag{83}$$

[1] [ρ can be treated as a dimensionless quantity, since the equation (77) is homogeneous.]

for β, which is a function of q. Here,

$$\dot{u} = q_\nu \frac{\partial u}{\partial q_\nu} \equiv q_\nu \partial_\nu u = \frac{\partial u}{\partial \xi}, \qquad \xi = \ln q.$$

As a result, we get for u and ϕ the following set of equations:

$$\ddot{u} - u + \beta^2\left[3\sinh^2\frac{\phi}{2} + \frac{\dot{\phi}^2}{4}\right]u = \frac{\dot{\beta}}{\beta}(\dot{u} - u) \tag{84}$$

$$\ddot{\phi} + 2\frac{\dot{u}}{u}\dot{\phi} - 3\sinh\phi = 0. \tag{85}$$

For a constant β, the conservation law

$$\partial_\xi E = 0$$

is fulfilled;

$$E = \dot{u}^2 - u^2 + \beta^2\left[3\sinh^2\frac{\phi}{2} - \frac{\dot{\phi}^2}{4}\right]u^2. \tag{86}$$

The term $(\dot{\beta}\beta)(\dot{u} - u)$ is of the order of g^2, and can almost always be neglected.

The asymptotic behaviour (74) of the Green's function in the limit $q^2 \to \infty$ ($\beta \to 1$) corresponds to

$$u \to Cq^2 \quad , \quad \phi \to \mathrm{i}\pi .$$

The chiral-invariant solution corresponds to $\phi \equiv \mathrm{i}\pi$. Close to the value $\phi = \mathrm{i}\pi$ (i.e., $\phi = \mathrm{i}\pi + \tilde{\phi}$), we have, at large q^2,

$$\frac{\dot{u}}{u} = \sqrt{1 + 3\beta^2}. \tag{87}$$

Hence, (85) is an oscillator equation, with damping if q^2 is increasing, and with acceleration if q^2 is decreasing:

$$\ddot{\tilde{\phi}} + 3\tilde{\phi} = -2\sqrt{1 + 3\beta^2}\,\dot{\tilde{\phi}}. \tag{88}$$

This means that the chiral-invariant solution $\phi = \mathrm{i}\pi$ is unstable in an asymptotically free theory.

Let us consider in detail the equation for ϕ at negative q^2 values. Due to (80), $\phi = \mathrm{i}\psi$ is, in this case, purely imaginary, and (85) describes the motion of the particle as a function of the "time" ξ in a periodic field; the damping (or the acceleration) is defined by (87).

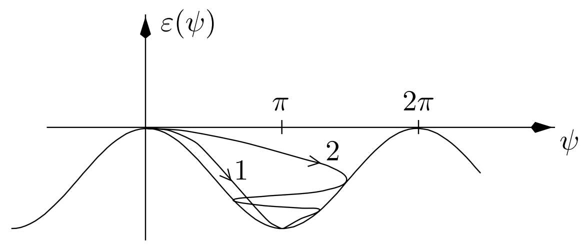


Fig. 2.

At $\xi \to \infty$ ($|q^2| \to \infty$), the particle is situated at one of the minima of the potential; it accelerates as ξ decreases.

The acceleration rate is defined by the parameters of the solution (74) in the limit $|q^2| \to \infty$. When q is decreasing, we have, generally speaking, two possible behaviours for the solution.

If $\dot{u}/u$ remains positive all the time, then the solution goes to infinity, and G develops a singularity at some Euclidean q. If instead the potential $\dot{u}/u$ changes sign, then the solution may again approach a minimum. In the latter case, it is easy to see that G^{-1} has a singularity at $q = 0$.

There is only one possibile way to avoid having a singularity in the Euclidean region including $q = 0$. We have to choose the parameters which determine the acceleration at $|q^2| \to \infty$ so that the particle approaches the maximum of the potential when $q \to 0$ ($\xi \to -\infty$). In order to find these exceptional solutions without singularities at $q^2 < 0$, it is natural to start solving the equation at $q = 0$ by fixing the behaviour of the solution at $\xi \to -\infty$. The solution of (84) and (85) corresponding to a maximum (e.g., $\psi = 0$) at $q \to 0$ is

$$\mathrm{i}\psi = \frac{q}{m_c}\,, \quad u = Z^{-1}(0)q\,. \tag{89}$$

In the case of such a solution, the constant E in (86) equals zero, and

$$\frac{\dot{u}}{u} = \sqrt{1 + \beta^2\left[3\sin^2\frac{\psi}{2} - \frac{\dot{\psi}^2}{4}\right]}\,. \tag{90}$$

Inserting (90) into (85) we obtain one nonlinear equation; it can be analyzed easily for arbitrary q^2 values. The solution we are interested in contains essentially one parameter m_c, which can be related to the renormalized fermion mass. The parameter Z^{-1} is irrelevant; it defines the normalization of the Green's function at $q = 0$, and can be chosen as 1. The solution of the equation leads to the unambiguous determination of the asymptotic parameters of the Green's function (74) by m_c and by the parameter λ_{QCD} of the strong interaction[2] which enters $\beta(q) = 1 - C_F\alpha(q^2)/\pi$.

As was mentioned before, the spontaneous breaking of chiral symmetry corresponds to an asymptotic behaviour of G^{-1}, in which $m = 0$. The existence of such a solution requires a connection between m_c and λ_{QCD}. The renormalized fermion mass, as well as the other condensate parameters, are then determined by the strong interaction parameter λ_{QCD}.

Let us consider this solution in detail. Starting from the point $\psi = 0$ at $\xi \to -\infty$, the solution will, obviously, reach the minimum of the potential either monotonically (if the damping is strong enough) or in an oscillating way. In our case, the damping depends on the value of β. If g is small, β is close to unity, the damping is strong, and the solution has a monotonic behaviour. By decreasing β, the solution may become an oscillating one. To obtain the value of β at which the oscillation begins, there is no need to solve the equation for all values of q. It will be sufficient to investigate the solution in the region where ψ becomes close to π; here $\psi = \pi + \tilde{\psi}$, and $\tilde{\psi}$ satisfies the equation (88). The solution can be written in the form

$$\tilde{\psi} = \mathrm{e}^{-p\xi} C \cos\left(\sqrt{2 - 3\beta^2}\xi + \delta\right), \qquad p = \sqrt{1 + 3\beta^2}\,. \tag{91}$$

It oscillates if

$$\beta^2 < \frac{2}{3}\,; \quad g_c \equiv 1 - \sqrt{\frac{2}{3}} \;<\; g \;<\; 1 + \sqrt{\frac{2}{3}}\,. \tag{92}$$

Oscillations in ψ mean that the mass term in (80),

$$m(q) \;\simeq\; \mathrm{i}\frac{\tilde{\psi}}{2}\,q\,, \tag{93}$$

starts to oscillate. By solving the equation for bound states we can check that at $g > g_c$ bound states appear with wave functions behaving like (91); this result coincides with the behaviour of the solution of the Dirac equation in the field of a point-like static charge Ze when $Z > 137$. The simplest example for such bound states are Goldstone states the wave functions of which, as we shall see in the next section, are proportional to $m(q)$.

We have found the oscillations and the critical value $g_c = 1 - \sqrt{2/3}$ using the assumption that g is a constant. In reality, g depends on q (see Fig. 1) and the situation is somewhat more complicated; it is reminiscent of the case of the equation for a critical charge of finite radius. In the region of small q values, where $g(q)$ is close to a constant $g(q) \approx g(0)$, we can consider ψ as an independent variable, and q^2 as a function of ψ. It can be shown that for $g(0)$ satisfying the condition (92), there are two regions, $0 < \psi < \pi - \psi_c$, and $\pi + \psi_c < \psi < 2\pi$ in the q^2, ψ plane (Fig. 3)

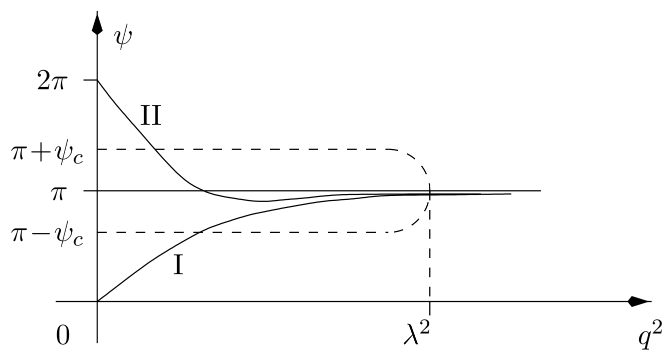


Fig. 3.

where the solution $\psi(q)$ is monotonic, and there is a region $\pi - \psi_c < \psi < \pi + \psi_c$ where the solution oscillates. The value of ψ_c is determined by the equality

$$\sin^2\frac{\psi_c}{2} = \left(\frac{2}{3} - \beta^2\right)\sqrt{\frac{1 + 3\beta^2}{1 - \beta^2}}\,\frac{1}{1 + \sqrt{(1 + 3\beta^2)(1 - \beta^2)}}\,, \tag{94}$$

[2] [λ_{QCD} is the momentum scale where the coupling in Fig. 1 becomes of order unity or, more precisely, when it reaches the critical value (see below)]

which exists if $\beta^2 < 2/3$. Considering β as a function of q^2 in (94) and taking into account that $\beta^2 \to 1$ at $|q^2| \to \infty$, we obtain a region bounded by the dashed curve and the vertical axis where the solution oscillates. There are no oscillations if $|q^2| > \lambda^2_{\rm QCD}$ ($\beta^2(\lambda^2_{\rm QCD}) = 2/3$). Due to (74) and (75), we can write, in the region $|q^2| \gg \lambda^2_{\rm QCD}$,

$$\frac{\mathrm{i}}{2}(\psi - \pi) = \frac{m_0}{q}\left(\frac{\alpha}{\alpha_0}\right)^{\gamma_m} + \frac{\nu_1^3}{q^3}\left(\frac{\alpha}{\alpha_0}\right)^{\gamma_1}. \tag{95}$$

If $\beta^2 > 2/3$, the solution which equals $\mathrm{i}\psi = q/m_c$ at $q \to 0$ transforms monotonically into (95), with $m_0 \neq 0$. If $\beta^2 < 2/3$, $\psi(\lambda^2_{\rm QCD})$ and $\dot\psi(\lambda^2_{\rm QCD})$ start to oscillate as functions of m_c, and m_0 can turn into zero at a certain m_c value. This means that we have a solution corresponding to broken chiral symmetry.

If $m_0 = 0$, there exist also a large number of solutions depending on the parameters ν_1 and ν_2. With the same sign of ν_1, we can have different solutions corresponding to the curves I and II, which, in the limit $q \to 0$, reach $\psi = 0$ and $\psi = 2\pi$, respectively. The solutions $\psi(0) = 2\pi$ correspond to smaller values ν_1, m_c.

Let us consider the solutions of types I and II in detail for complex q – this is justified since G^{-1} has to satisfy the requirements of analyticity and unitarity. We will show that both solutions have singularities at positive real q^2 values. The solutions are chosen in such a way that they are regular as $q \to 0$ and have no singularities at $q^2 < 0$. Due to the analyticity of the equations, the behaviour of the solution for $q^2 > 0$ can be found by solving (85) and (90) with the same boundary conditions at $q \to 0$.

If β is fixed, the equations (84) – (86) can be rewritten in a simpler form. We denote

$$\frac{\dot u}{u} = p(\phi).$$

Then

$$\frac{\partial p}{\partial \phi} = -\beta\sqrt{p^2 + 3\beta^2\sinh^2\frac{\phi}{2} - 1} \tag{96}$$

$$\dot\phi = \frac{2}{\beta}\sqrt{p^2 + 3\beta^2\sinh^2\frac{\phi}{2} - 1} \tag{97}$$

with the boundary condition $p = 1$ at $\phi = 0$ for a type-I solution.
[Equation (97) solves (86) for $E = 0$. Equation (96) follows from (97), and

$$\dot p = 1 - p^2 - \beta^2\left(\frac{\dot\phi^2}{4} + 3\sinh^2\frac{\phi}{2}\right),$$

which is equivalent to (84) for $\dot\beta = 0$.]
The phase diagram corresponding to (96) is shown in Fig. 4, where the solid line represents the solution of the equation

$$p^2 = 1 - 3\beta^2\sinh^2\frac{\phi}{2}. \tag{98}$$

The function $p = p(\phi)$ is shown by the dashed line in Fig. 4.

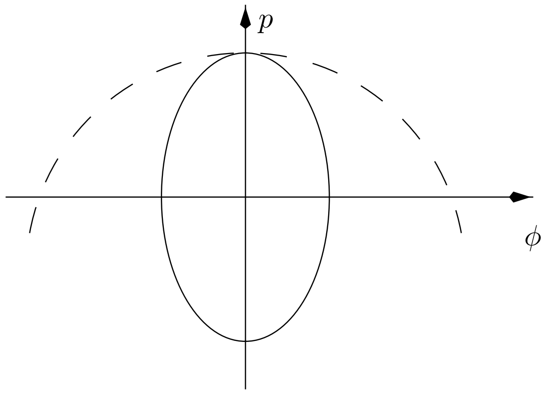


Fig. 4.

The ϕ dependence of p at $\phi \to \infty$ is different for $\beta > 1/2$ and $\beta < 1/2$. In both cases, ϕ approaches infinity at finite ξ values. At $\beta > 1/2$, $\phi \to \infty$, we have

$$\frac{\partial p}{\partial \phi} = \beta p, \qquad p = -\frac{C}{2}\mathrm{e}^{\beta\phi}; \qquad C > 0\,, \tag{99}$$

and at $\beta < 1/2$, $\phi \to \infty$,

$$p = -2\beta^2 C_1 \mathrm{e}^{\frac{\phi}{2}}, \qquad C_1 \equiv \frac{1}{2}\sqrt{\frac{3}{1 - 4\beta^2}}\,. \tag{100}$$

The equation (97) enables us to find ξ as a function of ϕ:

$$\xi = \xi^* - \frac{\beta}{2}\int_\phi^\infty \frac{\mathrm{d}\phi'}{\sqrt{p^2(\phi) + 3\beta^2\sinh^2\frac{\phi}{2} - 1}}\,. \tag{101}$$

The integral in (101) converges in both cases ($\beta > 1/2$ and $\beta < 1/2$); because of this, ϕ goes to infinity at a finite $\xi = \xi^*$ ($q \to m^*$). Near $q = m^*$ at $\beta > 1/2$ ($g < 1/2$) we have

$$u = u_0\sqrt{1 - \frac{q}{m^*}}\quad,\quad \mathrm{e}^{-\frac{\phi}{2}} = \left\{C\left(1 - \frac{q}{m^*}\right)\right\}^{\frac{1}{2\beta}}, \tag{102}$$

and, consequently,

$$G^{-1}(q) = Z_0^{-1}\left\{\left(1 - \frac{q}{m^*}\right)^{\frac{1}{\beta}}\frac{q + \hat q}{2} + \left(\frac{1}{C}\right)^{\frac{1}{\beta}}\frac{q - \hat q}{2}\right\}. \tag{103}$$

If $\beta < 1/2$ ($g > 1/2$),

$$u = u_0\left(1 - \frac{q}{m^*}\right)^{2\beta^2}\quad,\quad e^{-\frac{\phi}{2}} = C_1\left(1 - \frac{q}{m^*}\right), \tag{104}$$

and thus

$$G^{-1}(q) = Z_0^{-1}\left\{\left(1 - \frac{q}{m^*}\right)^{2\beta+1}\frac{q + \hat q}{2} + \left(1 - \frac{q}{m^*}\right)^{2\beta-1}\frac{1}{C_1^2}\frac{q - \hat q}{2}\right\}. \tag{105}$$

It can be shown that the relation between the quantity m_c and the position of the singularity at $q = m^*$ of the Green's function can be written as

$$\ln \frac{m^*}{m_c} = \int_0^\infty \mathrm{d}\phi \left[\frac{\beta}{2\sqrt{p^2(\phi) + 3\beta^2 \sinh^2 \frac{\phi}{2} - 1}} - \frac{2}{\sinh 2\phi} \right]. \quad (106)$$

The formulas (103) and (105) define the behaviour of $G^{-1}(q)$ near the singularity for a solution of type I.

In order to obtain the behaviour of G^{-1} near the singularity for a solution of type II, it is sufficient to notice that at $q \to 0$, such a solution has the form

$$\phi = 2\pi \mathrm{i} - \frac{q}{m'_c}. \quad (107)$$

The replacement of ϕ by $\phi + 2\pi\mathrm{i}$ changes only the sign of G^{-1}. Replacing q/m_c by $-q/m'_c$ and solving (96) and (97) for $q > 0$, we will find ϕ to be negative, and near the singularity, ϕ will go to $-\infty$. This means the following: If the first term $G_+^{-1}(q)$ of the expression

$$G^{-1}(q) = G_+^{-1}(q)\frac{1}{2}\left(1 + \frac{\hat{q}}{q}\right) + G_-^{-1}(q)\frac{1}{2}\left(1 - \frac{\hat{q}}{q}\right) \quad (108)$$

equals zero for the type-I solution at $q = m^*$, then G_-^{-1} is zero for the type-II solution at $q = m'^*$.

If $q^2 > m^{*2}$, both solutions become complex:

$$\phi = -\frac{1}{\beta} \ln \left[C \left(\frac{q}{m^*} - 1 \right) \right] + \frac{\mathrm{i}\pi}{\beta} \quad , \quad \beta > \frac{1}{2}$$
$$\phi = -2 \ln \left[C_1 \left(\frac{q}{m^*} - 1 \right) \right] + 2\mathrm{i}\pi \quad , \quad \beta < \frac{1}{2}. \quad (109)$$

The trajectories of $\phi(q)$ at $0 < q < \infty$ in the complex plane ϕ are shown in Fig. 5 for the type-I and -II solutions [for $\beta > 1/2$].

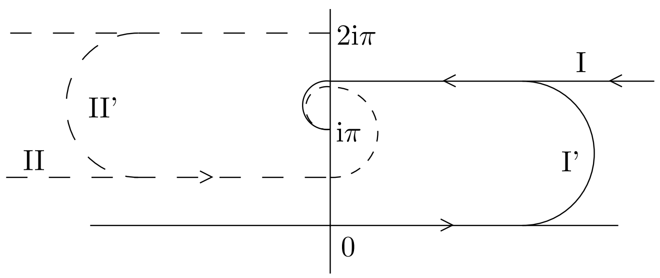


Fig. 5.

The type-I solution has a remarkable feature: $\mathrm{Im}(\phi) > \pi$ for any $q > m^*$ values. Hence, taking $\phi = \phi_1 + \mathrm{i}\phi_2$, the imaginary part of $m(q)$ in (80),

$$\mathrm{Im}(m(q)) = q\,\mathrm{Im}\left(\frac{\cosh \frac{1}{2}(\phi_1 + i\phi_2)}{\sinh \frac{1}{2}(\phi_1 + i\phi_2)} \right) = \frac{-q \sin \phi_2}{2|\sinh \frac{1}{2}(\phi_1 + i\phi_2)|^2}, \quad (110)$$

turns out to be positive. At the same time, $\mathrm{Im}(m(q))$ for the type-II solution is an oscillating function; this can lead to a contradiction of the unitarity condition for the Green's function.

For complex q values, $G^{-1}(q)$ has no singularities. If we move in the complex plane along an arbitrary ray [from the origin to infinity], the trajectory of $\phi(q)$ does not approach infinity (curves I' and II').

The main result of this section is the following: In the framework of the equation for the fermion Green's function (73) there exist solutions corresponding to broken symmetry, provided $g(q)$ has an asymptotically free behaviour and $g(0) > 1 - \sqrt{2/3}$. These solutions behave, at $q^2 \to \infty$, as

$$G^{-1}(q) = Z \left[\frac{\nu_1^3}{q^2} - \hat{q} \left(1 - \frac{\nu_2^4}{q^4} \right) \right]. \quad (111)$$

The expression (111) has a mass term decreasing at infinity.

6 Axial current conservation and Goldstone states

If a fermion Green's function corresponds to symmetry breaking, it is natural to expect the existence of Goldstone-type bound states. This expectation is connected with the belief that if $m_0 = 0$, the axial current has to be conserved. This is, however, not necessarily true; because of divergences in the theory, a leakage of the current into the region of the ultraviolet cutoff is possible. A typical example of this phenomenon is the anomaly. But even in a nonanomalous case, it is not obvious whether the current conservation is implied by the equation for the Green's function or is imposed as a condition on the solution of the equation. In order to clarify this, let us consider the equation for the bound state ϕ, supposing that it is a pseudoscalar.

$$\partial^2 \phi(p,q) = g(q) \left\{ A_\mu(q_2)\partial_\mu \phi(p,q) + \partial_\mu \phi(p,q)\tilde{A}_\mu(q_1) - A_\mu(q_2)\phi(p,q)\tilde{A}_\mu(q_1) \right\}. \quad (112)$$

This equation has to have a solution decreasing for large q^2 at $p^2 = 0$. It is easy to see that there indeed exists such a solution. At $p = 0$, the equation for ϕ is an equation for the variation of a function which satisfies the equation for G^{-1}. If this variation is taken in the form $\phi = C\{\gamma_5, G^{-1}\}$, ϕ obviously satisfies the equation and decreases at $|q^2| \to \infty$ as

$$\phi \;\to\; C\frac{2\nu_1^3}{q^2}\gamma_5\,. \quad (113)$$

However, this does not mean that we have particles with $p^2 = 0$. Indeed, the equation (112) has a solution decreasing at $|q^2| \to \infty$ for any p^2 values. The reason for this is that the equation is highly degenerate. It has a solution of the form

$$\phi = O_1 G^{-1}(q_1) + G^{-1}(q_2) O_2\,, \quad (114)$$

where O_1 and O_2 are any combinations of Dirac matrices with coefficients not depending on q. This can be checked by directly substituting (114) into (112) and using the equation for G^{-1}. The reason for this degeneracy is the invariance of the equation (112) under Lorentz transformations, under all discrete symmetries of the Dirac equation and, for $g = const$, even scale invariance and translational invariance in momentum space. If, for example, $O_1 = O_2 = \gamma_5$, then, due to the Ward identity,

$$p_\mu \Gamma^5_\mu(p,q) = \gamma_5 G^{-1}(q_1) + G^{-1}(q_2)\gamma_5 \,, \qquad (115)$$

$$\phi = \gamma_5 G^{-1}(q_1) + G^{-1}(q_2)\gamma_5 \qquad (116)$$

is the divergence of the axial current. Hence, if (112) is satisfied for $p_\mu \Gamma^5_\mu$, it has to have the solution (115).

In order to show that (112) has a decreasing solution at $|q^2| \to \infty$ for any p value, we first notice that in Euclidean space, this equation has the structure of the Schrödinger equation

$$(-\partial^2 + U)\Psi = \varepsilon\Psi, \qquad (117)$$

in four dimensions at $\varepsilon = 0$, with a potential depending on q, spin variables, and the external vector p. For such a potential, the total four-dimensional angular momentum is not conserved; only its projection μ onto p_ν is. An equation of this type always has nonsingular solutions with incoming waves of given μ, e.g., considering an incoming wave with $\mu = 0$ for the pseudoscalar ϕ, we will have a solution

$$\phi_0 = \gamma_5 C_0 + \quad \text{decreasing scattered waves}$$

at $q^2 \to \infty$. Or, taking an incoming wave of the form $\gamma_5 p_\mu \gamma_\mu \equiv \gamma_5 \hat{p}$, we will have

$$\phi_1 = \gamma_5 \hat{p} + \quad \text{decreasing scattered waves.}$$

Suppose we found the solution ϕ_1. Then, due to the fact that for $q \to \infty$ the solution (115) behaves as

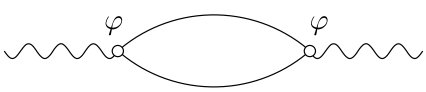
$$\gamma_5 G^{-1}(q_1) + G^{-1}(q_2)\gamma_5 \;\to\; Z^{-1}\left(\gamma_5\hat{p} + \frac{2\gamma_5\nu^3}{q^2}\right), \qquad (118)$$

we will find that

$$\phi = Z^{-1}\phi_1 - \gamma_5 G^{-1}(q_1) - G^{-1}(q_2)\gamma_5 \;\to\; -\frac{2Z^{-1}\gamma_5\nu^3}{q^2} \qquad (119)$$

is decreasing, with $|q^2| \to \infty$ at any p.

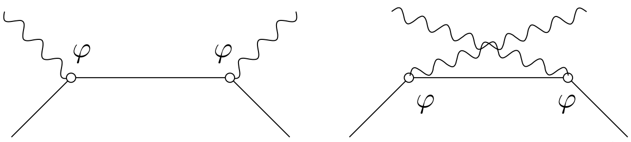
By stating that the equation for bound states has a solution at any value of p, we do not mean that there are no bound states; we mean only that the mass of the bound state has to be calculated independently. The most natural way to do this is to calculate the self-energy of the state

$\Sigma(p) =$  (120)

and to solve the equation

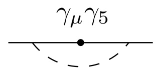
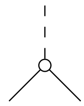
$$\Sigma(p) = 0\,. \qquad (121)$$

Alternatively, one can calculate the forward Compton scattering of the bound state on a fermion,

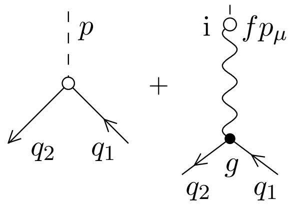
 , (122)

and then integrate over the distribution of fermions in the vacuum. But this way, we will never get a massless Goldstone state. The reason for this is that (120) and (122) contradict the condition of axial current conservation.

Let us consider the condition for current conservation in detail. If we introduce $\tilde{\Gamma}^5_\mu$ as a set of diagrams

$\tilde{\Gamma}^5_\mu = \gamma_\mu\gamma_5 +$  $+ \ldots =$  (123)

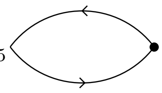
with a massive fermionic Green's function, it will not satisfy the Ward identity. However, the Ward identity will be satisfied by the sum of $\tilde{\Gamma}^5_\mu$ and of the Goldstone contribution:

$\Gamma^5_\mu =$  , (124)

namely $[p = q_1 - q_2]$,

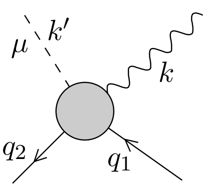
$$p_\mu \Gamma^5_\mu = p_\mu \tilde{\Gamma}^5_\mu - \mathrm{i}fg \;=\; \gamma_5 G^{-1}(q_1) + G^{-1}(q_2)\gamma_5\,. \qquad (125)$$

Here g is the Yukawa coupling of a Goldstone boson to a fermion [and f is related to g via the fermion-loop diagram]

$\mathrm{i}fp_\mu = \gamma_\mu\gamma_5$  $g.$ (126)

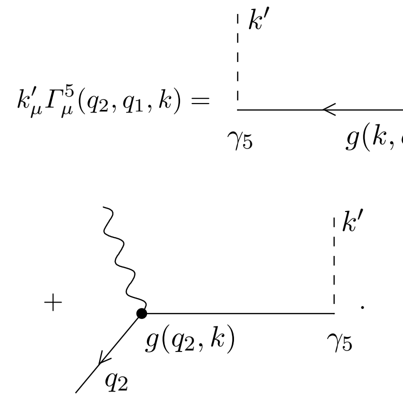
Knowing $p_\mu \tilde{\Gamma}^5_\mu$, we can define the Yukawa coupling g by (125) and (126). Let us apply the operator $-\partial^2 + U$ to (125). We find that gf satisfies the equation (112), since $p_\mu \tilde{\Gamma}^5_\mu$ and the right-hand side of (125) satisfy the same equation. But the existence of (124) implies that the mass of the Goldstone boson has to be zero.

In order to clarify the situation with Compton scattering, let us consider the Ward identity for the amplitude

$\Gamma^5_\mu(q_2, q_1, k) =$  (127)

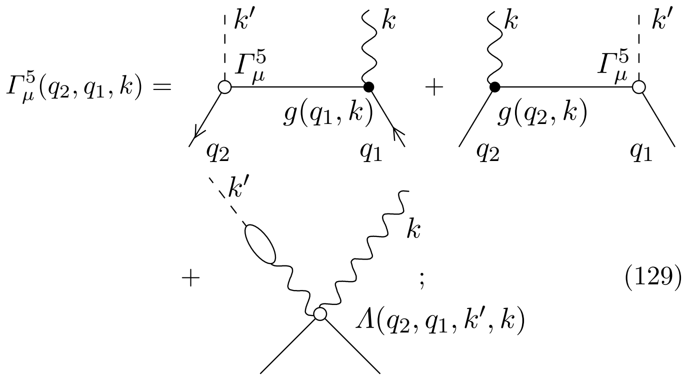
where the dashed line with the momentum k' corresponds to the axial current, the wavy line with momentum k

to the Goldstone state, and the other two lines to the fermions. The Ward identity for this amplitude is



(128)

To fulfil (128), $\Gamma^5_\mu(q_2,q_1,k)$ has to be



; (129)

here Γ^5_μ is defined by (124). Using (124) and (128), we obtain

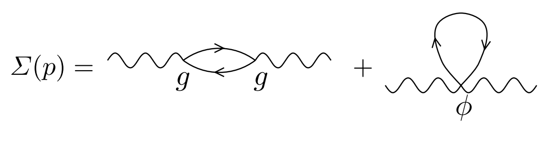
$$\mathrm{i} f \Lambda = \gamma_5 g(k,q_1) + g(q_2,k)\gamma_5 \,. \tag{130}$$

At large q_1^2, q_2^2 values we will have

$$g = \mathrm{i}\frac{2\gamma_5}{f}\frac{\nu^3}{q^2}\ ; \quad \Lambda = \frac{4\nu^3}{f^2 q^2}. \tag{131}$$

This means that, as in the case of asymptotically nonfree theories, the Goldstone–fermion scattering amplitude does not depend on the momentum of the Goldstone boson; it decreases only as a function of fermion virtuality. Under these circumstances, it is obvious that even in an asymptotically free theory, the Goldstone boson has a kind of point-like structure.

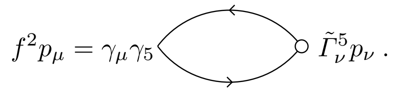
Amplitudes for the interactions of many Goldstones with fermions can be found in an analogous way, and have the same property [the property of being independent of the Goldstone momenta]. The self-energy of the Goldstone state is now different from (120). It contains two terms



(132)

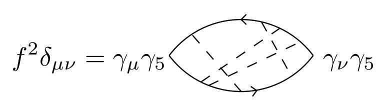
and equals p^2 at small p^2.

As we already have said, in this approach the Ward identity becomes the definition of the Yukawa coupling g (the wave function of the Goldstone boson) through $p_\mu \Gamma^5_\mu$ which has a clear diagrammatic meaning. The equation contains the amplitude f of the transition between the Goldstone and the axial current. The expression (126) for $f p_\mu$ is highly symbolic, because it contains overlapping divergences. In order to write a sensible expression we can use the same procedure as in Sect. 2 where we calculated the polarization operator of the photon. Applying the Ward identity, we can write, instead of (126),



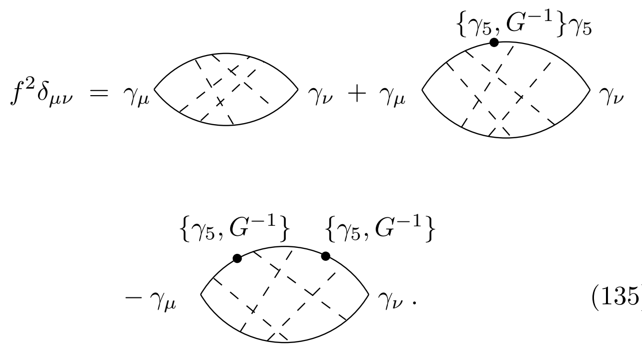
(133)

Differentiating (133) over p, we will get, for $p = 0$,



(134)

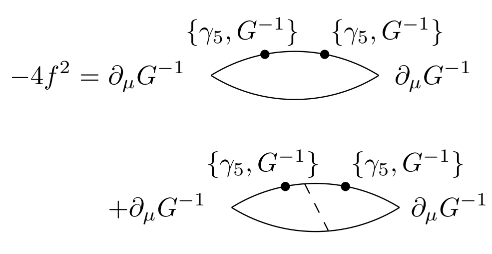
[This diagrammatic expression contains exact fermion propagators and bare vertices.] If we want to get rid of the γ_5-s, we have to commute γ_5 with the Green's functions and the interaction vertices along one of the fermionic lines. Doing this, we obtain

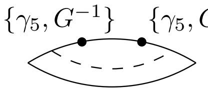


(135)

Due to the conservation of the vector current, the first two terms in (135) are zero at $p = 0$. The first one is just the photon polarization operator at $p = 0$, the second one is the amplitude for the decay of a zero-momentum scalar into two zero-momentum photons.

The last term looks like the zero-momentum pseudoscalar-photon scattering amplitude which also has to be zero. This, however, is not true, because it does not contain all the necessary diagrams. Observing that this term does not contain overlapping divergences we can write



$$+\,\partial_\mu G^{-1}$$  $$\partial_\mu G^{-1} + \ldots \quad (136)$$

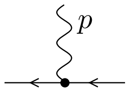
In the zeroth order in α/π,

$$f^2 = \frac{1}{4}\int \frac{\mathrm{d}^4 q}{(2\pi)^4 \mathrm{i}}\,\mathrm{Tr}\left(\{\mathrm{i}\gamma_5, G^{-1}\}\,G\,\{\mathrm{i}\gamma_5, G^{-1}\}\right.$$
$$\left.\times\, G\,A_\mu(q)\,A_\mu(q)\right). \quad (137)$$

7 Flavour singlet and flavour nonsinglet Goldstone states: The U(1) problem

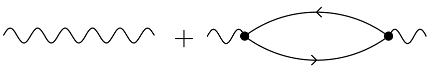
Up to now we have discussed the Goldstone states in a relatively abstract way, without fixing the concrete asymptotically free theory. In real QCD, we have quarks with different flavours and there is a difference between flavour singlet and flavour nonsinglet states. In order to clarify the picture it will be useful to describe the Goldstone state in a different way.

The previous discussion shows that the Goldstone states in asymptotically free and nonfree theories are rather similar. Therefore it is natural to try to introduce the Goldstone boson in the usual way, as a point-like state, and to see how this state will interact. In the usual discussion of a Goldstone particle, it is we supposed that there is a point-like pseudoscalar interaction between this particle and a fermion with the pseudovector coupling

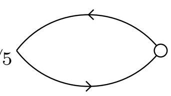


$$= \hat{p}\gamma_5\frac{1}{f_0}. \quad (138)$$

This interaction induces the radiative correction to the propagator $D(p^2)$ of the pseudoscalar

$$D(p^2) =$$  $$+\cdots\;, \quad (139)$$

where the wavy line describes the bare pseudoscalar Green's function D_0. If the fermion is massless and the axial current is conserved, this pseudoscalar will not interact; its self-energy,

$$\Sigma(p^2) = \frac{1}{f_0^2}\,p_\mu\gamma_\mu\gamma_5$$  $$p_\nu\tilde{\Gamma}^5_\nu \;\equiv\; p_\mu p_\nu \sigma_{\mu\nu}, \quad (140)$$

is equal to zero. If, due to symmetry breaking, the fermion becomes massive, it starts to interact, and acquires a self-energy different from zero. By using the diagrammatic definition of $\tilde{\Gamma}^5_\mu$ [see (133)], we will find

$$\Sigma(p^2) = \frac{p^2 f^2}{f_0^2}, \quad (141)$$

where f is the same amplitude for the Goldstone-current transition as what was discussed in the previous section. Hence,

$$D(p^2) = \frac{f_0^2}{D_0^{-1} f_0^2 - p^2 f^2}. \quad (142)$$

In the limit $f_0 \to 0$, we will have

$$D(p^2) = -\frac{f_0^2}{p^2 f^2}. \quad (143)$$

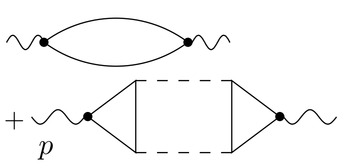
The pseudovector interaction between the Goldstone and the fermion is $\frac{1}{f} p_\mu \tilde{\Gamma}^5_\mu$, with a pseudovector coupling $1/f$ defined by the fermion mass. The limiting procedure $f_0 \to 0$ can be understood if we accept that the interaction responsible for symmetry breaking changes the fermion-vacuum fluctuations not only at finite momenta, but also near the ultraviolet cutoff. This change in the fermion vacuum fluctuations is responsible for the leakage of the axial current in [from] the region of finite momenta; it can produce the driving term for the Goldstone state, recovering the current conservation.

In general, the pseudovector coupling has a disadvantage compared to the pseudoscalar coupling that we discussed before: It looks unrenormalizable. But in the case of flavour nonsinglet states, it can always be replaced by the pseudoscalar coupling with the help of the trivial relation $[p = q_1 - q_2]$

$$-\hat{p}\gamma_5 = \big(\hat{q}_2 - m(q_2)\big)\gamma_5 + \gamma_5\big(\hat{q}_1 - m(q_1)\big)$$
$$+ m(q_2)\gamma_5 + \gamma_5 m(q_1)\,, \quad (144)$$

which leads to the Ward identity (125) for pseudoscalar coupling. If we include the emission and the absorption of Goldstone bosons inside the diagram, then in the process of this replacement, a point-like amplitude appears which corresponds to the quark interaction with many Goldstone bosons. Nevertheless, I believe it is possible to prove that, due to the decrease of these amplitudes as functions of quark virtuality, the theory is renormalizable.

Due to the anomaly, the case of a flavour singlet current is very different, even on the level of the Goldstone Green's function. In this case the corresponding polarization operator Σ will contain not only the quark loop which we have discussed, but also a gluonic contribution

$$\Sigma(p^2) =$$  $$. \quad (145)$$

The triangle diagram $f_{\mu\nu}$ present in (145) was calculated many years ago by Adler, Bell and Jackiw [3]:

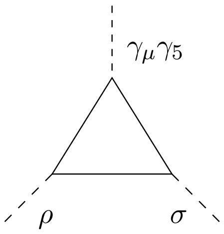
$$f_{\mu\nu} = \frac{\alpha}{\pi}\varepsilon_{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma}. \quad (146)$$

With this expression for $f_{\mu\nu}$, $\Sigma(p^2)$ still has the form (141) but it will be quadratically divergent, and

$$f^2 = f_{q\bar{q}}^2 + \left(\frac{\alpha}{\pi}\right)^2 \Lambda_{\mathrm{UV}}^2 \to \infty\,, \quad (147)$$

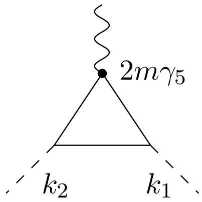
where Λ_{UV} is the ultraviolet cutoff. This means that Goldstone particles exist in the anomalous case, but they are decoupled from all physical states. At the same time, the Ward identity still makes sense, because the product gf

does not depend on f. Nevertheless, the concrete form of the Ward identity will change. The reason for this is again the Adler–Bell–Jackiw anomaly [3]. For the triangle diagram

$$\Delta^\mu_{\rho\sigma} =$$


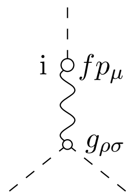
$$, \tag{148}$$

the replacement of the pseudovector coupling by the pseudoscalar coupling gives an incorrect result: Instead of the correct expression

$$p_\mu \Delta^\mu_{\rho\sigma} =$$


$$+\frac{\alpha}{\pi}\varepsilon_{\rho\sigma\delta\gamma}k_{2\delta}k_{1\gamma}, \tag{149}$$

which was obtained in [3], we get just the first term. We can try to write the Ward identity using (149), but this turns out not to be necessary. The reason is that, in this approach, the axial current $\Gamma^\mu_{\rho\sigma}$ between gluonic states, defined symbolically by the relation

$$\Gamma^\mu_{\rho\sigma} = \tilde{\Gamma}^\mu_{\rho\sigma} +$$


$$\tag{150}$$

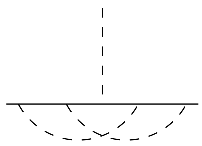
where the term $\tilde{\Gamma}^\mu_{\rho\sigma}$ is defined diagrammatically, and $g_{\rho\sigma} = (p_\nu/\mathrm{i}f)\tilde{\Gamma}^\nu_{\rho\sigma}$ is just the transverse part of $\tilde{\Gamma}^\mu_{\rho\sigma}$:

$$\Gamma^\mu_{\rho\sigma} = \tilde{\Gamma}^\mu_{\rho\sigma} - \frac{p_\mu p_\nu}{p^2}\tilde{\Gamma}^\nu_{\rho\sigma}. \tag{151}$$

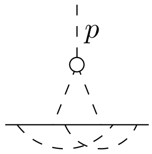
The axial current between quark states $\tilde{\Gamma}^\mu$ can be written as

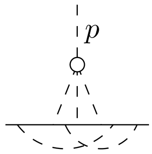
$$\tilde{\Gamma}^\mu = \tilde{\tilde{\Gamma}}^\mu(q_2, q_1) + \tilde{\Gamma}^\mu_g(q_2, q_1). \tag{152}$$

Here

$$\tilde{\tilde{\Gamma}} =$$

$$+\dots \tag{153}$$

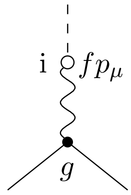
is the same set of diagrams as in the nonsinglet case, and $\tilde{\Gamma}^\mu_g$ is the axial current of gluons

$$\tilde{\Gamma}^\mu_g =$$


$$+$$


$$+\dots \quad . \tag{154}$$

In the same way, the Goldstone–quark interaction can also be divided into two parts. In the first part, we can replace the pseudovector coupling by the pseudoscalar coupling. The second part is the longitudinal part of $\tilde{\Gamma}^\mu_g$. Consequently,

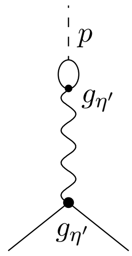
$$\Gamma^\mu = \tilde{\tilde{\Gamma}}^\mu +$$


$$+\tilde{\Gamma}^\mu - \frac{p_\mu p_\nu}{p^2}\tilde{\Gamma}^\nu, \tag{155}$$

and the Ward identity can be written in the form

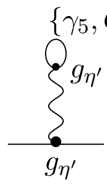
$$p_\mu \tilde{\tilde{\Gamma}}^\mu - \mathrm{i}fg = \gamma_5 G^{-1}(q_1) + G^{-1}(q_2)\gamma_5. \tag{156}$$

This expression enables us to answer the question: What happens with particles like η', which would be Goldstone states if we did not take into account that they can decay into two gluons? Let us consider the contribution of the massive pseudoscalar flavour singlet particle η' to the Ward identity (156). The divergence $p_\mu\tilde{\tilde{\Gamma}}^\mu$ has a pole corresponding to η':

$$p_\mu \tilde{\tilde{\Gamma}}^\mu =$$


$$= \mathrm{i}\,p^2 f_{\eta'}\frac{1}{\mu^2 - p^2}\,g_{\eta'}\,. \tag{157}$$

The Yukawa coupling of the Goldstone state also has a pole:

$$-\mathrm{i}\,fg =$$


$$= \{\gamma_5, G^{-1}(q)\}$$

$$g_{\eta'}\frac{1}{\mu^2 - p^2}\,g_{\eta'}\,. \tag{158}$$

The right-hand side of (156), however, has no poles. This condition can be satisfied if

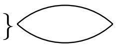
$$\mu^2 f_{\eta'} = \{\mathrm{i}\gamma_5, G^{-1}(q)\}$$

$$g_{\eta'}\,. \tag{159}$$

The same Ward identity (156) [substituting (157) and (158), and using (159)] gives, at $p^2 = 0$,

$$f_{\eta'}g_{\eta'} = \{\mathrm{i}\gamma_5, G^{-1}(q)\}\,, \tag{160}$$

from which it follows that μ^2 for η' is[3]

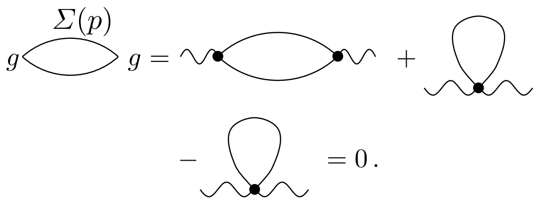
$$\mu^2_{\eta'} = \frac{\{\mathrm{i}\gamma_5, G^{-1}(q)\} \; [\text{loop}] \; \{\mathrm{i}\gamma_5, G^{-1}(q)\}}{f^2_{\eta'}}$$

$$= g_{\eta'}$$

$$g_{\eta'}\,. \tag{161}$$

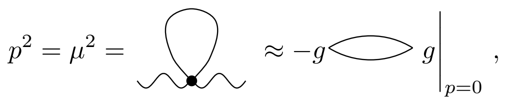
[3] [The overall sign in the following equation differs from that in the original manuscript.]

This means that η' has acquired a mass due to the transition into a Goldstone boson, which itself is decoupled.

It is interesting to notice that at relatively small μ^2, when the comparison between Ward identities for different values of p^2 ($p^2 = \mu^2$, $p^2 = 0$) makes sense, (161) gives us the same result as (120) (without the additional point-like term that is present in the π-meson case (132)). Indeed, for (120) we can write



The sum of the first two terms is equal to p^2 (as in (132)) and therefore



(162)

in agreement with (161)[4]. We see that the subtraction term in the π-meson self-energy, reflecting the quasi-local structure of the pion, disappears in the case of η'. In this sense, η' is a normal bound state without a point-like structure.

The approach we presented here for the resolution of the U(1) problem is technically very close to the approach developed by Veneziano [4], but the underlying physics differs essentially. In Veneziano's approach big long-range fluctuations are responsible for the η' mass. In our approach, η' is a normal $q\bar{q}$ bound state with no local structure which would be responsible for its small mass if it were a Goldstone state. This local structure is destroyed by the decay on hard gluons.

8 QCD with massive quarks

We have discussed in detail an asymptotically free theory with massless fermions. We came to the conclusion that in order to obtain the correct spectrum of Goldstone particles, axial current conservation has to be imposed on the theory. QCD, however, contains massive quarks and the same spectrum of massive quasi-Goldstone particles (the pseudoscalar octet) as the theory with massless quarks. The problem is how to impose the condition of axial current conservation on a theory which obviously does not conserve the axial current. Strictly speaking, I don't know how to do this. For our real world, however, there is a natural possibility to solve the problem.

In the real world, QCD is part of the standard model describing strong, electromagnetic, and weak interactions. In the standard model, all particles are supposed to be intrinsically massless and their masses appear as the result of symmetry breaking due to some kind of Higgs mechanism with or without elementary Higgs particles. The possibility of such a mechanism is guaranteed by the conservation of the left-handed SU(2)-current j^a_μ. For the matrix elements Γ^μ_a of this current between any two quarks with momenta q_2, q_1, we have the Ward identity $[p = q_1 - q_2]$

$$p_\mu \Gamma^\mu_a(q_2, q_1) = G^{-1}(q_2)\frac{1-\gamma_5}{2}\frac{\tau_a}{2} - \frac{1+\gamma_5}{2}\frac{\tau_a}{2}G^{-1}(q_1)\,. \tag{163}$$

In the case of massive fermions, $p_\mu \Gamma^\mu_a$ contains the contribution of three Goldstone bosons responsible for the masses of $W^\pm$ and Z^0:

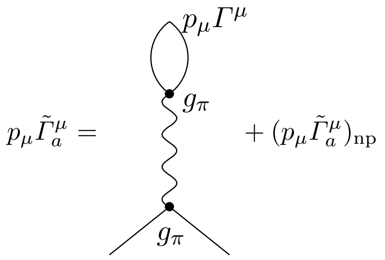
$$p_\mu \Gamma^\mu_a = p_\mu \tilde{\Gamma}^\mu_a - f g^a\,, \quad \left[g^a = \frac{\tau_a}{2}\, g\right]. \tag{164}$$

For large q^2 values, G^{-1} contains a massive term $Z^{-1}m_0$. Hence, at large q^2 we have

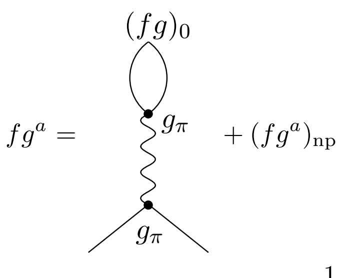
$$(fg^a)_0 = \frac{1}{4}\left\{\tau_a\gamma_5, m_0 Z^{-1}\right\} + \frac{1}{4}[\tau_a, m_0 Z^{-1}]. \tag{165}$$

At small q^2 of the order of the QCD scale, fg is defined by the total quark mass $m(q)$, which, for the light quark, is much larger than m_0. Using the relations (163)–(165) we can calculate the masses of the pseudoscalar octet in the same way as we did for η'.

Suppose there is a bound state of light quarks which is a pseudoscalar particle (the π meson) with a finite mass μ. The pole corresponding to this particle will contribute to both terms in (164). With this pole contribution, $p_\mu \tilde{\Gamma}^\mu_a$ and fg can be written in the form [with np standing for "non-pion" or "non-pole"]

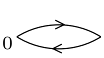


$$= f_\pi p^2 \,\frac{1}{\mu^2 - p^2}\, g^a_\pi \;+\; (p_\mu \tilde{\Gamma}^\mu_a)_{\rm np}\,, \tag{166}$$



$= (fg^a)_0$  $g^b_\pi \,\dfrac{1}{\mu^2 - p^2}\, g^b_\pi + (fg^a)_{\rm np}\,.$ (167)

As in the case of η', the condition for the pole cancellation gives us the value of μ^2:

$\mu^2 \cdot \delta_{ab} = \dfrac{1}{f_\pi}(fg^a)_0$  $g^b_\pi.$ (168)

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[4] [Notice however that the expressions (161) and (162) have different signs; see the previous footnote.]

At $p^2 = 0$, we have

$$f_\pi g_\pi^a + (fg^a)_{\rm np} = \frac{1}{4}\left\{\tau_a\gamma_5, \hat{m}(q)Z^{-1}\right\} + \frac{1}{4}\left[\tau_a, \hat{m}Z^{-1}\right]. \quad (169)$$

[Here $\hat{m}$ is the light-quark mass matrix, $\hat{m} = m_s + (1/2)(m_{\rm u} - m_{\rm d})\tau_3$, with $m_s = (1/2)(m_{\rm u} + m_{\rm d})$.] The Yukawa coupling g_π of the bound state has to decrease at large q^2. Therefore we have to identify $(fg)_{\rm np}$ with $(fg)_0$. Then

$$f_\pi g_\pi^a = \frac{\tau_a}{2}\gamma_5\, m_s(q) Z^{-1}\,, \quad (170)$$

where $m_s(q)$ is the isotopically invariant part of the quark mass term behaving at $q \to \infty$ as ν^3/q^2. Finally, $(fg)_0 \propto m_0$ is defined by weak interactions with Goldstone states, and $f_\pi g_\pi \propto m_s$ is determined by pion exchange. The conclusion of these considerations is that in the case of massive quarks the conservation of the left-handed SU(2) current can play the same role for the calculation of the coupling and the masses of the flavour nonsinglet pseudoscalar particles as does the conservation of the axial current in the massless case. The result is that the masses become different from zero, while the Yukawa coupling remains the same at least for small m_0 values.

The expression (168) for μ^2 can be also obtained without reference to the current conservation, just by calculating $\Sigma(p)$ as it is defined in (132) for the quark mass $m_s + m_0$, keeping only the term linear in m_0. This is in agreement with the idea that m_0 affects the mass of the Goldstone state but not its wave function, an idea which always has been the common belief.

It is not trivial that if we calculate μ^2 by using (168) or (132), we get a logarithmic divergence. For the π-meson mass, the main divergent part is

$$m_\pi^2 = \frac{3}{4\pi^2}\frac{1}{f_\pi^2}\int_{\nu_0^2}^{\infty}\frac{{\rm d}(q^2)}{q^2}\, m_0(q)\,\nu^3(q) \quad (171)$$

(here ν_0 is of the order of $\lambda_{\rm QCD}$).

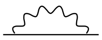
In the standard model, the behaviour of m_0 and ν^3 can be calculated in a fantastically wide region of q^2: From 1 GeV up to the scale where one of the couplings of the standard model (the U(1) coupling g_1, the Yukawa coupling h, or the coupling of the self-interaction of the Higgs particles, λ) becomes of the order of unity. If the U(1) coupling g_1 is the first to become large, which seems natural, this scale is $\Lambda_{\rm UV} \simeq 10^{38}$ GeV.

At $q^2 > \Lambda_{\rm UV}^2$, the behaviour of m_0 and ν^3 is unknown. Because of this, m_π is not calculable in principle, and has to be considered as an arbitrary parameter. If we assume, however, that at q^2 larger than $\Lambda_{\rm UV}$, $m_0(q)$ and $\nu^3(q)$ vanish, we will be able to calculate m_π and, surprisingly, this calculation gives a reasonable value for m_π with $\Lambda_{\rm UV} \simeq 10^{38}$ GeV [2]. The expression (171) for m_π^2 is also in agreement with the naive expression

$$m_\pi^2 = \frac{2m_0}{f_\pi^2}\langle\bar{\Psi}\Psi\rangle.$$

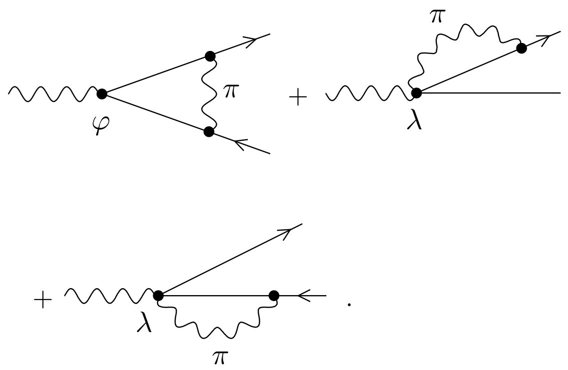
The important difference is that now $\langle\bar{\Psi}\Psi\rangle$ is determined not only by strong but also by weak interactions: It goes to infinity if the weak interaction is removed.

9 The pion contribution to the equation for light-quark Green's functions

From the previous discussion, it is obvious that the small-mass pion contribution has to be included in the equation for the light-quark Green's function. Fortunately this is very easy to do. Having in mind that now the diagrams contain not only the gluon contribution but also the emission and the absorption of pions, we will find that, as before, the main contribution to $\partial^2 G^{-1}$ comes from the simplest diagram  with the coupling $\{{\rm i}\gamma_5, G^{-1}\}$ at zero momentum, instead of the gluon coupling $\partial_\mu G^{-1}$. This leads to the following equation for the Green's function:

$$\partial^2 G^{-1}(q) = g\partial_\mu G^{-1}(q)G(q)\partial_\mu G^{-1}(q) - \{{\rm i}\gamma_5, G^{-1}\}G(q)\{{\rm i}\gamma_5, G^{-1}\}\frac{3}{16\pi^2 f_\pi^2}\,. \quad (172)$$

The equation for bound states (27) has to be changed also. The correction comes from the diagrams

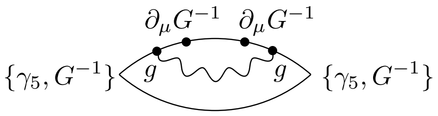


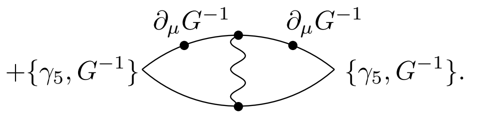
Instead of (27), we will have

$$\begin{aligned}
&\partial^2\phi^a(p,q) = \\
&= g(q)\{A_\nu(q_2)\partial_\nu\phi^a(p,q) + \partial_\nu\phi^a(p,q)\tilde{A}_\nu(q_1) \\
&- A_\nu(q_2)\phi^a(p,q)\tilde{A}_\nu(q_1)\} \\
&+ \frac{1}{4\pi^2 f_\pi^2}\Big[\{{\rm i}\gamma_5, G^{-1}(q_2)\}G(q_2)\frac{\tau_b}{2} \\
&\quad\times \phi^a\frac{\tau_b}{2}G(q_1)\{{\rm i}\gamma_5, G^{-1}(q_1)\} \\
&- \lambda G(q_1)\{{\rm i}\gamma_5, G^{-1}(q_1)\}\frac{\tau_a}{2} - \{{\rm i}\gamma_5, G^{-1}(q_2)\}G(q_2)\frac{\tau_a}{2}\lambda\Big].
\end{aligned} \quad (173)$$

Here λ is the emission amplitude of the zero-momentum pion in the transition of the bound state to the $q\bar{q}$ pair. This amplitude has to be defined by the axial current conservation. There is another important quantity in this equation, namely, f_π. In Sect. 6, we have written an explicit expression for f_π^2, including only the gauge field contribution and ignoring the pion contribution. Now we include the pion contribution in the equation for the Green's function and, to be self-consistent, we have to do the same for f_π^2. I was not able to carry this out in any order in the

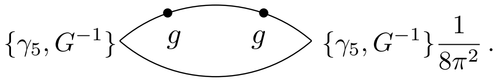
Yukawa coupling, but in the first order in g^2, the equation (136) is correct if one adds the diagrams





(174)

As we see, the gluonic correction of the order of α/π and the pionic correction of the order of g^2 have the same diagrammatic structure. In order to estimate the value of these corrections, let us take just the contributions of zero-momentum gluons and pions to them. It can be shown that the contribution of zero-momentum gluons cancels in the last two diagrams of (136). Zero-momentum pion contribution comes only from the first diagram of (174). Transferring the differentiation from the fermionic line to the pionic line in this diagram, we obtain the contribution of zero-momentum pions in the form



(175)

The expression for f_π^2, which includes the zero-momentum pion contribution, is

$$8f_\pi^2 = \int \frac{\mathrm{d}^4 q}{(2\pi)^4 i}\ \mathrm{Tr}\left(\{\mathrm{i}\gamma_5, G^{-1}\}G\{\mathrm{i}\gamma_5, G^{-1}\}G\, A_\mu\, A_\mu\right)$$
$$+\ \frac{1}{8\pi^2 f_\pi^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4 \mathrm{i}}\,\mathrm{Tr}\left(\{\mathrm{i}\gamma_5, G^{-1}\}G\right)^4 . \qquad (176)$$

This gives us an understanding of the scale of possible pion contributions. It is interesting to note that (176) is not an expression in terms of the Green's functions, but an equation for f_π.

In the next paper, it will be shown that the pion contribution changes essentially the structure of the equation for the Green's function. The new equation has a solution corresponding to the confined quark. At the same time, the symmetry-breaking solution will not necessarily survive (at least if $g(0)$ is large).

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